Subjective Logic

Draft, 18 February 2013

Audun Jøsang

University of Oslo

Web: http://folk.uio.no/josang/
Email: josang@mn.uio.no

Preface

Subjective logic is a type of probabilistic logic that allows probability values to be expressed with degrees of uncertainty. The idea of probabilistic logic is to combine the strengths of logic and probability calculus, meaning that it has binary logic's capacity to express structured argument models, and it has the power of probabilities to express degrees of truth of those arguments. The idea of subjective logic is to extend probabilistic logic by also expressing uncertainty about the probability values themselves, meaning that it is possible to reason with argument models in presence of uncertain or incomplete evidence.

In this manuscript we describe the central elements of subjective logic. More specifically, we first describe the representations and interpretations of subjective opinions which are the input arguments to subjective logic. We then describe the most important subjective logic operators. Finally, we describe how subjective logic can be applied to trust modelling and for analysing Bayesian networks.

Subjective logic is directly compatible with binary logic, probability calculus and classical probabilistic logic. The advantage of using subjective logic is that real world situations can be more realistically modelled, and that conclusions more correctly reflect the ignorance and uncertainties that necessarily result from partially uncertain input arguments.

Contents

Chapter 1

Introduction

In standard logic, propositions are considered to be either true or false, and in probabilistic logic the arguments are expressed as a probability in the range $[0, 1]$. However, a fundamental aspect of the human condition is that nobody can ever determine with absolute certainty whether a proposition about the world is true or false, or determine the probability of something with 100% certainty. In addition, whenever the truth of a proposition is assessed, it is always done by an individual, and it can never be considered to represent a general and objective belief. This indicates that important aspects are missing in the way standard logic and probabilistic logic capture our perception of reality, and that these reasoning models are more designed for an idealised world than for the subjective world in which we are all living.

The expressiveness of arguments in a reasoning model depends on the richness in the syntax of those arguments. Opinions used in subjective logic offer significantly greater expressiveness than binary or probabilistic values by explicitly including degrees of uncertainty, thereby allowing an analyst to specify *"I don't know"* or *"I'm indifferent"* as input argument. Definitions of operators used in a specific reasoning model are based on the argument syntax. For example, in binary logic the AND, OR and XOR operators are defined by their respective truth tables which traditionally have the status of being axioms. Other operators, such as MP (Modus Ponens), MT (Modus Tollens) and other logical operators are defined in a similar way. In probabilistic logic the corresponding operators are simply algebraic formulas that take continuous probability values as input arguments. It is reasonable to assume that binary logic TRUE corresponds to probability 1, and that FALSE corresponds to probability 0. With this correspondence binary logic simply is an instance of probabilistic logic, or equivalently on can say that probabilistic logic is a generalisation of binary logic. More specifically there is a direct correspondence between binary logic operators and probabilistic logic algebraic formulas, as e.g. specified in Table 1.

Binary Logic		**Probabilistic Logic**	
AND:	$x \wedge y$	Product:	$p(x \wedge y) = p(x)p(y)$
OR:	$x \vee y$	Coproduct:	$p(x \vee y) = p(x) + p(y) - p(x)p(y)$
XOR:	$x \underline{\vee} y$	Inequivalence	$p(x \not\equiv y) = p(x)(1 - p(y)) + (1 - p(x))p(y)$
MP:	$\{x \to y,\ x\} \Rightarrow y$	Deduction:	$p(y\|x) = p(x)p(y\|x) + p(\overline{x})p(y\|\overline{x})$
MT:	$\{x \to y,\ \overline{y}\} \Rightarrow \overline{x}$	Abduction:	$p(x\tilde{\|}y) = \frac{a(x)p(y\|x)}{a(x)p(y\|x)+a(\overline{x})p(y\|\overline{x})}$ $p(x\tilde{\|}\overline{y}) = \frac{a(x)p(\overline{y}\|x)}{a(x)p(\overline{y}\|x)+a(\overline{x})p(\overline{y}\|\overline{x})}$ $p(x\overline{\|}y) = p(y)p(x\tilde{\|}y) + p(\overline{y})p(x\tilde{\|}\overline{y})$

Table 1.1: Correspondence between binary logic and probabilistic logic operators

The notation $p(y\|x)$ means that the probability of y is derived as a function of the conditionals $p(y|x)$ and $p(y|\overline{x})$ as well as the antecedent $p(x)$. The parameter $a(x)$ represents the base rate of x. The symbol "$\not\equiv$" represents inequivalence, i.e. that x and y have different truth values.

MP (Modus Ponens) corresponds to conditional deduction, and MT (Modus Tollens) corresponds to conditional abduction in probability calculus. The notation $p(y\|x)$ for conditional deduction denotes the output probability of

y conditionally deduced from the input conditional $p(y|x)$ and $p(y|\overline{x})$ as well as the input argument $p(x)$. Similarly, the notation $p(x\overline{\|}y)$ for conditional abduction denotes the output probability of x conditionally abduced from the input conditional $p(y|x)$ and $p(y|\overline{x})$ as well as the input argument $p(y)$.

For example, consider the case of MT where $x \rightarrow y$ is TRUE and y is FALSE, which translates into $p(y|x) = 1$ and $p(y) = 0$. Then it can be observed from the first equation that $p(x|y) \neq 0$ because $p(y|x) = 1$. From the second equation it can be observed that $p(x|\overline{y}) = 0$ because $p(\overline{y}|x) = 1 - p(y|x) = 0$. From the third equation it can finally be seen that $p(x\overline{\|}y) = 0$ because $p(y) = 0$ and $p(x|\overline{y}) = 0$. From the probabilistic expressions it can thus be abduced that $p(x) = 1$ which translates into x being FALSE, as MT dictates.

The power of probabilistic logic is the ability to derive logic conclusions without relying on axioms of logic, only on principles of probability calculus.

Probabilistic logic was first defined by Nilsson [21] with the aim of combining the capability of deductive logic to exploit the structure and relationship of arguments and events, with the capacity of probability theory to express degrees of truth about those arguments and events. Probabilistic logic can be used to build reasoning models of practical situations that are more general and more expressive than reasoning models based on binary logic.

A serious limitation of probabilistic logic, and binary logic alike, is that it is impossible to express input arguments with degrees of ignorance as e.g. reflected by the expression *"I don't know"*. An analyst who is unable to provide any reliable value for a given input argument can be tempted or even forced to set a value without any evidence to support it. This practice will generally lead to unreliable conclusions, often described as the "garbage in - garbage out" problem. In case the analyst wants to express *"I don't know the truth values of* x_1 *and* x_2*"* and needs to derive $p(x_1 \wedge x_2)$, then probabilistic logic does not offer a good model. Kleene's three-valued logic also does not offer an adequate model. Kleene's logic dictates value UNDEFINED for $(x_1 \wedge x_2)$ when x_1 and x_2 are defined as UNDEFINED. However, in case of an arbitrarily large number of variables x_i that are all UNDEFINED, Kleene's logic would still dictate UNDEFINED for $(x_1 \wedge \cdots \wedge x_n)$, whereas the correct value should be FALSE. A simple example illustrates why this is so. Assume the case of flipping a fair coin multiple times. An observer's best guess about whether the first outcome will be *heads* might be expressed as *"I don't know"* which in 3-valued logic would be expressed as UNDEFINED, but an observer's guess about whether the first n outcomes will all be *heads*, when n is arbitrarily large or even infinite, should intuitively be expressed as FALSE, because the likelihood that a infinite series of outcomes will only produce *heads* becomes infinitesimally small.

The additivity principle of classical probability requires that the probabilities of mutually disjoint elements in a state space add up to 1. This requirement makes it necessary to estimate a probability value for every state, even though there might not be a basis for it. On other words, it prevents us from explicitly expressing ignorance about the possible states, outcomes or statements. If somebody wants to express ignorance about the state x as *"I don't know"* this would be impossible with a simple scalar probability value. A probability $P(x) = 0.5$ would for example mean that x and $\overline{x}$ are equally likely, which in fact is quite informative, and very different from ignorance. Alternatively, a uniform probability density function would more closely express the situation of ignorance about the outcome of the outcome.

Arguments in subjective logic are called *"subjective opinions"* or *"opinions"* for short. An opinion can contain degrees of uncertainty in the sense of *"uncertainty about probability estimates"*. The uncertainty of an opinion can be interpreted as ignorance about the truth of the relevant states, or as second order probability about the first order probabilities.

The subjective opinion model extend the traditional belief function model in the sense that opnions take base rates into account whereas belief functions ignore base rates. Belief theory has its origin in a model for upper and lower probabilities proposed by Dempster in 1960. Shafer later proposed a model for expressing belief functions [24]. The main idea behind belief theory is to abandon the additivity principle of probability theory, i.e. that the sum of probabilities on all pairwise disjoint states must add up to one. Instead belief theory gives observers the ability to assign so-called belief mass to the powerset of the state space. The main advantage of this approach is that ignorance, i.e. the lack of evidence about the truth of the states, can be explicitly expressed e.g. by assigning belief mass to the whole state space. Shafer's book [24] describes various aspects of belief theory, but the two main elements are 1) a flexible way of expressing beliefs, and 2) a conjunctive method for combining belief functions, commonly known as Dempster's Rule, which in subjective logic is called the belief constraint operator. Subjective opinions represent a generalisation of belief functions because subjective opinions include base rates, and in that sense have a richer expressiveness than belief functions.

Defining logic operators on subjective opinions is normally quite simple, and a relatively large set of practical logic operators exists. This provides the necessary framework for reasoning in a large variety of situations where input arguments can be incomplete or affected by uncertainty. Subjective opinions are equivalent to Dirichlet and Beta probability density functions. Through this equivalence subjective logic provides a calculus for reasoning with probability density functions.

This manuscript provides a general introduction to subjective logic. Different but equivalent representations of subjective opinions are presented together with their interpretation. This allows uncertain probabilities to be seen from different angles, and allows an analyst to define models according to the formalisms that they are most familiar with, and that most naturally represents a specific real world situation. Subjective logic contains the same set of basic operators known from binary logic and classical probability calculus, but also contains some non-traditional operators which are specific to subjective logic.

The advantage of subjective logic over traditional probability calculus and probabilistic logic is that real world situations can be modeled and analysed more realistically. The analyst's partial ignorance and lack of information can be taken explicitly into account during the analysis, and explicitly expressed in the conclusion. When used for decision support, subjective logic allows decision makers to be better informed about uncertainties affecting the assessment of specific situations and future outcomes.

Chapter 2

Elements of Subjective Opinions

2.1 Motivation

A fundamental aspect of the human condition is that nobody can ever determine with absolute certainty whether a proposition about the world is true or false. In addition, whenever the truth of a proposition is expressed, it is always done by an individual, and it can never be considered to represent a general and objective belief. These philosophical ideas are directly reflected in the mathematical formalism and belief representation of subjective logic.

Explicit expression of uncertainty is the main motivation for subjective logic and for using opinions as arguments. Uncertainty comes in many flavours, and a good taxonomy is described in in [25]. In subjective logic, the uncertainty relates to probability estimates. For example, let the probability estimate of a future event x be expressed as $P(x) = 0.5$. In case this probability estimate represents the perceived likelihood of obtaining heads when flipping a fair coin, then it would be natural to represent it as an opinion with zero uncertainty. In case the probability estimate represents the perceived likelihood that there is life on a planet in a specific solar system, then it would be natural to represent it as an opinion with considerable uncertainty. The probability estimate of an event is thus separated from the certainty/uncertainty of the probability value. With this explicit representation of uncertainty subjective logic can be applied in case of events that have totally uncertain probability estimates, i.e. where the analyst is ignorant about the likelihood of possible events. This is possible by including the degree of uncertainty about probability estimates as an explicit parameter in the input opinion arguments. This uncertainty is then taken into account during the analysis and explicitly represented in the output conclusion.

2.2 Flexibility of Representation

There can be multiple syntactic representa

An opinion is a composite function consisting of belief masses, uncertainty mass and base rates which are described separately below. An opinion applies to a frame, also called a state space, and can have an attribute that identifies the belief owner. The belief masses are distributed over the frame or over the reduced powerset of the frame in a sub-additive fashion, meaning that the sum of belief masses normally is less than one.

An important property of opinions is that they are equivalent Beta or Dirichlet probability density functions (pdf) under a specific mapping. This equivalence can be derived from the distribution of belief masses. The sum of belief masses depends on the amount of evidence underlying the pdf in such a way that an infinite amount of evidence corresponds to an additive belief mass distribution (i.e. the sum is equal to one), and a finite amount of evidence corresponds to a subadditive belief mass distribution (i.e. the sum is less that one). In practical situations the amount of evidence is never infinite, so that practical opinions should always be subadditive. The basic elements of subjective opinions are defined in the sections below.

2.3 The Reduced Powerset of Frames

Let X be a frame of cardinality k. The powerset of X, denoted as $\mathcal{P}(X)$ or equivalently as 2^X, has cardinality 2^k and contains all the subsets of X, including X and $\emptyset$. In subjective logic, the belief mass is distributed over the reduced powerset denoted as $\mathcal{R}(X)$. More precisely, the reduced powerset $\mathcal{R}(X)$ is defined as:

$$\mathcal{R}(X) = 2^X \setminus \{X, \emptyset\} = \{x_i \mid i = 1 \ldots k,\ x_i \subset X\}\ , \tag{2.1}$$

which means that all proper subsets of X are elements of $\mathcal{R}(X)$, but X itself is not in $\mathcal{R}(X)$. The emptyset $\emptyset$ is also not considered to be a proper element of $\mathcal{R}(X)$.

Let κ denote the cardinality of $\mathcal{R}(X)$, i.e. $\kappa = |\mathcal{R}(X)|$. Given the frame cardinality $k = |X|$, then we have $\kappa = (2^k - 2)$, i.e. there are only $(2^k - 2)$ elements in the reduced powerset $\mathcal{R}(X)$ because it is assumed that X and $\emptyset$ are not elements of $\mathcal{R}(X)$. It is practical to define the first k elements of $\mathcal{R}(X)$ as having the same index as the corresponding singletons of X. The remaining elements of $\mathcal{R}(X)$ should be indexed in a simple and logical way. The elements of $\mathcal{R}(X)$ can be grouped in classes according to the number of singletons from X that they contain. Let j denote the number of singletons in the elements of a specific class, then we will call it "class j". By definition then, all elements belonging to class j have cardinality j. The actual number of elements belonging to each class is determined by the Choose Function $\mathrm{C}(\kappa, j)$ which determines the number of ways that j out of κ singletons can be chosen. The Choose Function, equivalent to the binomial coefficient, is defined as:

$$\mathrm{C}(\kappa, j) = \binom{\kappa}{j} = \frac{\kappa!}{(\kappa - j)!\ j!} \tag{2.2}$$

Within a given class each element can be indexed according to the order of the lowest indexed singletons from X that it contains. As an example, Fig.2.1 below illustrates a frame X of cardinality $k = 4$ where the subset of 3 shaded singleton indicates a specific element of $\mathcal{R}(X)$.

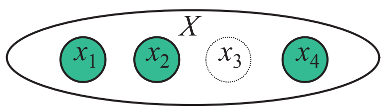


Figure 2.1: Example frame with 3 out of 4 selected singletons

The fact that this particular element contains 3 singletons defines it as a class-3 element, and the set of 3 specific singletons defines it as the element $\{x_1, x_2, x_4\}$. The two first selected singletons x_1 and x_2 have the smallest indexes that are possible to select, but the third selected singleton x_4 has the second smallest index that is possible to select. This particular element must therefore be assigned the second relative index in class 3. However, its absolute index depends on the number of elements in the smaller classes. Table 2.1 specifies the number of elements of classes 1 to 3, as determined by Eq.(2.2).

Class (element cardinality):	1	2	3
Number of elements of each class:	4	6	4

Table 2.1: Number of elements per class

Class 1 has 4 elements, and class 2 has 6 elements, which together makes 10 elements. Because the example of Fig.2.1 represents the 2nd relative index in class 3, its absolute index is $10 + 2 = 12$. Formally this is then expressed as $x_{12} = \{x_1, x_2, x_4\}$. To complete the example, Table 2.2 defines the index and class of all the elements of $\mathcal{R}(X)$ according to this scheme.

Class-1 elements are the original singletons from X, meaning that we can state the equivalence that $(x_i \in X) \Leftrightarrow$ (x_i is a class-1 element in $\mathcal{R}(X)$). The frame $X = \{x_1, x_2, x_3, x_4\}$ does not figure as an element of $\mathcal{R}(X)$ in Table 2.2 because excluding X is precisely what makes $\mathcal{R}(X)$ a reduced powerset. An element of $\mathcal{R}(X)$ that contains multiple singletons is called a *confusion element* because it represent a confusion of multiple singletons. In other words,

		Singleton selection per element													
Singletons	x_4				*			*		*	*		*	*	*
	x_3			*			*		*		*	*		*	*
	x_2		*			*			*	*		*	*		*
	x_1	*				*	*	*				*	*	*	
Element Index:		1	2	3	4	5	6	7	8	9	10	11	12	13	14
Element Class:		1				2						3			

Table 2.2: Index and class of elements of $\mathcal{R}(X)$ in case $|X| = 4$.

when an element is a non-singleton, or equivalently is not a class 1 element, then it is a confusion element in $\mathcal{R}(X)$. This is formally defined below.

Definition 1 (Confusion Element and Confusion Set) *Let X be a frame where $\mathcal{R}(X)$ is its reduced powerset. Every proper subset $x_j \subset X$ of cardinality $|x_j| \geq 2$ is a confusion element. The set of confusion elements $x_j \in \mathcal{R}(X)$ defines the* confusion set, *denoted by $\mathcal{C}(X)$.*

It is straightforward to prove that $\mathcal{R}(X) = X \cup \mathcal{C}(X)$. The cardinality of the confusion set is expressed as $|\mathcal{C}(X)| = \kappa - k$. In Sec.3.5 a description is provided of the degree of confusion in an opinion as a function of the belief mass assigned to confusion elements in $\mathcal{C}(X)$.

2.4 Belief Distribution over the Reduced Powerset

Subjective logic allows various types of belief mass distributions over a frame X. The distribution vector can be additive or subadditive, and it can be restricted to elements of X or it can include proper subsets of X. A belief mass on a proper subset of X is equivalent to a belief mass on an element of $\mathcal{R}(X)$. When the belief mass distribution is subadditive, the sum of belief masses is less than one, and the complement is defined as *uncertainty mass*. When the belief mass distribution is additive, there is no uncertainty mass. In general, the belief vector $\vec{b}_X$ specifies the distribution of belief masses over the elements of $\mathcal{R}(X)$, and the uncertainty mass denoted as u_X represents the uncertainty about the probability expectation value, as will be explained below. The subadditivity of the belief vector and the complement property of the uncertainty mass are expressed by Eq.(2.3) and Eq.(2.4) below.

$$\text{Belief sub-additivity: } \sum_{x_i \in \mathcal{R}(X)} \vec{b}_X(x_i) \leq 1 \;, \quad \vec{b}_X(x_i) \in [0,1] \tag{2.3}$$

$$\text{Belief and uncertainty additivity:} u_X + \sum_{x_i \in \mathcal{R}(X)} \vec{b}_X(x_i) = 1 \;, \quad \vec{b}_X(x_i), u_X \in [0,1] \tag{2.4}$$

There exists a direct correspondence between the bba function m used for representing belief masses in traditional belief theory, and the $\vec{b}_X$ and u_X functions used in subjective logic. The correspondence is defined such that $u_X = m(X)$ and $\vec{b}(x_i) = m(x_i), \ \forall x_i \in \mathcal{R}(X)$. The purpose of separating belief and uncertainty mass is motivated by the equivalent representation of Dirichlet and Beta probability density functions described below.

2.5 Base Rates over Frames

The concept of base rates is central in the theory of probability. Base rates are for example useful for default and for conditional reasoning. Traditional belief theory does not specify base rates. Without base rates however, there are many situations where belief theory does not provide an adequate model for expressing intuitive beliefs. This section specifies base rates for belief functions.and shows how it can be used for probability projections.

Given a frame of cardinality k, the default base rate of for each singleton in the frame is $1/k$, and the default base rate of a subset consisting of n singletons is n/k. In other words, the default base rate of a subset is equal to the number of singletons in the subset relative to the cardinality of the whole frame. A subset also has default *relative base rates* with respect to every other fully or partly overlapping subset of the frame.

However, in practical situations it would be possible and useful to apply base rates that are different from the default base rates. For example, when considering the base rate of a particular infectious disease in a specific population, the frame can be defined as {*"infected"*, *"not infected"*}. Assuming that an unknown person enters a medical clinic, the physician would *a priori* be ignorant about whether that person is infected or not before having assessed any evidence. This ignorance should intuitively be expressed as a vacuous belief function, i.e. with the total belief mass assigned to (*"infected"* ∪ *"not infected"*). The probability projection of a vacuous belief function using default base rate of 0.5 would dictate that the *a priori* probability of having the disease is 0.5. Of course, the base rate of diseases is normally much lower, and can be determined by relevant statistics from a given population.

The actual base rate can often be accurately estimated, as e.g. in the case of diseases within a population. Typically, data is collected from hospitals, clinics and other sources where people diagnosed with a specific disease are treated. The amount of data that is required to calculate a reliable base rate of the disease will be determined by some departmental guidelines, statistical analysis, and expert opinion about the data that it is truly reflective of the actual number of infections – which is itself a subjective assessment. After the guidelines, analysis and opinion are all satisfied, the base rate will be determined from the data, and can then be used with medical tests to provide a better indication of the likelihood of specific patients having contracted the disease [5].

Integrating base rates with belief functions provides a basis for a better and more intuitive interpretation of belief functions, facilitates probability projections from belief functions and provides a basis for conditional reasoning. When using base rates for probability projections, the share of contributing belief mass from subsets of the frame will be a function of this base rate function.

The base rate function is a vector denoted as $\vec{a}$ so that $\vec{a}(x_i)$ represents the base rate of the elements $x_i \in X$. The base rate function is formally defined below.

Definition 2 (Base Rate Function) Let X be a frame of cardinality k, and let $\vec{a}_X$ be the function from X to $[0,1]^k$ satisfying:

$$\vec{a}_X(\emptyset) = 0, \quad \vec{a}_X(x_i) \in [0,1] \quad \text{and} \quad \sum_{i=1}^{k} \vec{a}_X(x_i) = 1 \; . \tag{2.5}$$

Then $\vec{a}_X$ is a base rate distribution over X.

Two different observers can share the same base rate vectors. However, it is obvious that two different observers can also assign different base rates to the same frame, in addition to assigning different beliefs to the frame. This naturally reflects different views, analyses and interpretations of the same situation by different observers. Base rates can thus be partly objective and partly subjective.

Events that can be repeated many times are typically frequentist in nature, meaning that the base rates for these often can be derived from statistical observations. For events that can only happen once, the analyst must often extract base rates from subjective intuition or from analyzing the nature of the phenomenon at hand and any other relevant evidence. However, in many cases this can lead to considerable uncertainty about the base rate, and when nothing else is known, the default base rate of the singletons in a frame must be defined to be equally partitioned between them. More specifically, when there are k singletons in the frame, the default base rate of each element is $1/k$.

The difference between the concepts of subjective and frequentist probabilities is that the former can be defined as subjective betting odds – and the latter as the relative frequency of empirically observed data, where the subjective probability normally converges toward the frequentist probability in the case where empirical data is available [1]. The concepts of *subjective* and *empirical* base rates can be interpreted in a similar manner where they also converge and merge into a single base rate when empirical data about the population in question is available.

The usefulness of base rate function emerges from its application as the basis for probability projection. Because belief mass can be assigned to any subset of the frame it is necessary to also represent the base rates of such subsets. This is defined below.

Definition 3 (Subset Base Rates) Let X be a frame of cardinality k, and let $\mathcal{R}(X) = 2^X \setminus \{X, \emptyset\}$ be its reduced powerset of cardinality $\kappa = (2^k - 2)$. Assume that a base rate function $\vec{a}_X$ is defined over X according to Def.2.

Then the base rates of the elements of the reduced powerset $\mathcal{R}(X)$ are expressed according to the powerset base rate function $\vec{a}_{\mathcal{R}(X)}$ from $\mathcal{R}(X)$ to $[0,1]^{\kappa}$ defined below.

$$\vec{a}_{\mathcal{R}(X)}(\emptyset) = 0 \quad \text{and} \quad \vec{a}_{\mathcal{R}(X)}(x_i) = \sum_{\substack{x_j \in X \\ x_j \subseteq x_i}} \vec{a}_X(x_j)\ , \quad \forall x_i \in \mathcal{R}(X)\ . \tag{2.6}$$

Note that $x_j \in X$ means that x_j is a singleton in X, so that the subset base rate in Eq.(2.6) is the sum of base rates on singletons x_j contained in the element x_i. Trivially, it can be seen that when $x_i \in X$ then $\vec{a}_{\mathcal{R}(X)}(x_i) \equiv \vec{a}_X(x_i)$, meaning that $\vec{a}_{\mathcal{R}(X)}$ simply is an extension of $\vec{a}_X$. Because of this strong correspondence between $\vec{a}_{\mathcal{R}(X)}$ and $\vec{a}_X$ we will simply denote both base rate functions as $\vec{a}_X$. The base rate function has the same syntactic requirements as a traditional probability function, such as additivity, i.e. the sum of base rates over all mutually exclusive subsets equals one.

Because belief masses can be assigned to fully or partially overlapping subsets of the frame it is necessary to also derive relative base rates of subsets as a function of the degree of overlap with each other. This is defined below.

Definition 4 (Relative Base Rates) Assume X to be a frame of cardinality k where $\mathcal{R}(X) = 2^X \setminus \{X, \emptyset\}$ is its reduced powerset of cardinality $\kappa = (2^k - 2)$. Assume that a base rate function $\vec{a}_X$ is defined over X according to Def.3. Then the base rates of an element x_i relative to an element x_j is expressed according to the relative base rate function $\vec{a}_X(x_i/x_j)$ defined below.

$$\vec{a}_X(x_i/x_j) = \frac{\vec{a}_X(x_i \cap x_j)}{\vec{a}_X(x_j)}\ , \quad \forall\ x_i, x_j \in \mathcal{R}(X)\ . \tag{2.7}$$

Chapter 3

Opinion Classes and Representations

Subjective opinions express beliefs about the truth of propositions under degrees of uncertainty, and can indicate ownership (of the opinion) whenever required. A subjective opinion is normally denoted as ω^A_X where A is the opinion owner, also called the subject, and X is the target frame to which the opinion applies. An alternative notation is $\omega(A : X)$. There can be different classes of opinions, of which *hyper opinions* are the most general. *Multinomial opinions* and *binomial opinions* represent specif

The propositions of a frame are normally assumed to be exhaustive and mutually disjoint, and belief owners are assumed to have a common semantic interpretation of propositions. The belief owner (subject) and the propositions (object) are optional attributes of an opinion. The opinion itself is a composite function consisting of the belief vector $\vec{b}_X$, the uncertainty mass u_X and the base rate vector $\vec{a}_X$.

A few specific classes of opinions can be defined. In case of binary frames, the opinion is binomial. In case the frame is larger than binary and only singletons of X (i.e. class-1 elements of $\mathcal{R}(X)$) are focal elements, then it is called a *multinomial opinion*. In case the frame is larger than binary and there are focal elements of any class of $\mathcal{R}(X)$), then it is called a *hyper opinion*. Opinions can also be classified according to uncertainty. In case when $u_X > 0$ it is called an *uncertain opinion*. In case $u_X = 0$ it is called a *dogmatic opinion*.

More specific opinion classes can be defined, such as DH opinion (Dogmatic Hyper), UB Opinion (Uncertain Binomial) etc. The six main opinion classes defined in this way are listed in Table 3.1 below, and are described in more detail in the next section.

	Binomial Cardinality $\lvert X\rvert = 2$ $X = \mathcal{R}(X)$	**Multinomial** Cardinality $\lvert X\rvert > 2$ Focal elements $x \in X$	**Hyper** Cardinality $\lvert X\rvert > 2$ Focal elements $x \in \mathcal{R}(X)$
Uncertain $u > 0$	**UB opinion** Beta pdf	**UM opinion** Dirichlet pdf over X	**UH opinion** Dirichlet pdf over $\mathcal{R}(X)$
Dogmatic $u = 0$	**DB opinion** Scalar probability	**DM opinion** Probabilities on X	**DH opinion** Probabilities on $\mathcal{R}(X)$

Table 3.1: Opinion classes with equivalent probabilistic representations

The six entries in Table 3.1 also mention the equivalent probability representation of opinions, e.g. as Beta pdf, Dirichlet pdf or as a distribution of scalar probabilities over elements of X or $\mathcal{R}(X)$. This equivalence is explained in more detail in the following section.

The intuition behind using the term ”dogmatic” is that a totally certain opinion (i.e. where $u = 0$) about a real-world proposition can be seen as an extreme opinion. From a philosophical viewpoint nobody can ever be totally certain about anything in this world, so when it is possible to explicitly express degrees of uncertainty as with opinions, it can be seen as arrogant and extreme when somebody explicitly expresses a dogmatic opinion. This interpretation is confirmed when considering that a dogmatic opinion has an equivalent probability density function in the form of a singularity requiring an infinite amount of evidence. This does not mean that traditional probabilities should be interpreted as dogmatic, because their representation does not allow uncertainty to be expressed explicitly. Instead it can implicitly be assumed that there is some uncertainty associated with every probability estimate. One advantage of subjective logic is precisely that it allows explicit expression of this uncertainty.

The notation ω_X^A is traditionally used to denote opinions in subjective logic, where the subscript indicates the frame or proposition to which the opinion applies, and the superscript indicates the owner entity of the opinion. Subscripts can be omitted when it is clear and implicitly assumed to which frame an opinion applies, and superscripts can be omitted when it is irrelevant who the belief owner is.

Each opinion class will have an equivalence mapping to a type of Dirichlet or a Beta pdf (probability density function) under a specific mapping so that opinions can be interpreted as a probability density function. This mapping then gives subjective opinions a firm basis in notions from classical probability and statistics theory.

3.1 Binomial Opinions

3.1.1 Binomial Opinion Representation

Opinions over binary frames are called binomial opinions, and a special notation is used for their mathematical representation. A general n-ary frame X can be considered binary when seen as a binary partitioning consisting of one of its proper subsets x and the complement $\overline{x}$.

Definition 5 (Binomial Opinion) *Let $X = \{x, \overline{x}\}$ be either a binary frame or a binary partitioning of an n-ary frame. A binomial opinion about the truth of state x is the ordered quadruple $\omega_x = (b, d, u, a)$ where:*

b	(belief):	*the belief mass in support of x being true,*
d	(disbelief):	*the belief mass in support of x being false,*
u	(uncertainty):	*the amount of uncommitted belief mass,*
a	(base rate):	*the* a priori *probability in the absence of committed belief mass.*

These components satisfy $b + d + u = 1$ and $b, d, u, a \in [0, 1]$. The characteristics of various binomial opinion classes are listed below. A binomial opinion:

where $b = 1$	is equivalent to binary logic TRUE,
where $d = 1$	is equivalent to binary logic FALSE,
where $b + d = 1$	is equivalent to a traditional probability,
where $b + d < 1$	expresses degrees of uncertainty, and
where $b + d = 0$	expresses total uncertainty.

The probability projection of a binomial opinion on proposition x is defined by Eq.(3.1) below.

$$\mathrm{E}_x = b + au \tag{3.1}$$

Binomial opinions can be represented on an equilateral triangle as shown in Fig.3.1. A point inside the triangle represents a (b, d, u) triple. The belief, disbelief, and uncertainty-axes run from one edge to the opposite vertex indicated by the b_x axis, d_x axis and u_x axis labels. For example, a strong positive opinion is represented by a point towards the bottom right belief vertex. The base rate[1], is shown as a point on the base line, and the probability expectation, E_x, is formed by projecting the opinion point onto the base, parallel to the base rate director line. The opinion $\omega_x = (0.2,\ 0.5,\ 0.3,\ 0.6)$ with expectation value $\mathrm{E}_x = 0.38$ is shown as an example.

The class of binomial opinions where $u \geq 0$ is called UB opinion (Uncertain Binomial), whereas the opinion class where $u = 0$ is called DB opinion (Dogmatic Binomial). A DB opinion is equivalent to a classical scalar probability.

It can be seen that for a frame X of cardinality $k = 2$ a multinomial and a hyper opinion both have 3 degrees of freedom which is the same as for binomial opinions. Thus both multinomial and hyper opinions collapse to binomial opinions in case of binary frames.

In case the opinion point is located at one of the three vertices in the triangle, i.e. with $b = 1$, $d = 1$ or $u = 1$, the reasoning with such opinions becomes a form of three-valued logic that is compatible with Kleene logic [2].

[1] Also called *relative atomicity*

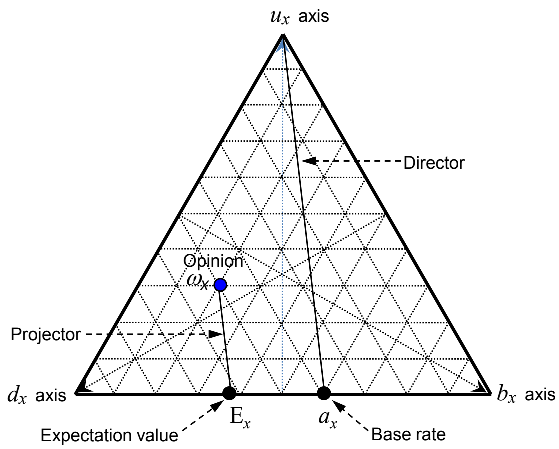


Figure 3.1: Opinion triangle with example opinion

However, the three-valued arguments of Kleene logic do not contain base rates, so that probability expectation values can not be derived from Kleene logic arguments.

In case the opinion point is located at the left or right bottom vertex in the triangle, i.e. with $b = 1$ or $d = 1$ and $u = 0$, the opinion is equivalent to boolean TRUE or FALSE, and is called an AB (Absolute Binomial) opinion. Reasoning with AB opinions is the same as reasoning in binary logic.

3.1.2 The Beta Binomial Model

A general UB opinion corresponds to a Beta pdf (probability density function) normally denoted as $\text{Beta}(p \,|\, \alpha, \beta)$ where α and β are its two evidence parameters. Beta pdfs are expressed as:

$$\begin{aligned} &\text{Beta}(p \,|\, \alpha, \beta) = \tfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \,, \\ &\text{where } 0 \le p \le 1,\ \alpha > 0,\ \beta > 0 \,, , \end{aligned} \tag{3.2}$$

with the restriction that the probability variable $p \neq 0$ if $\alpha < 1$, and $p \neq 1$ if $\beta < 1$.

Let r denote the number of observations of x, and let s denote the number of observations of $\overline{x}$. The α and β parameters can be expressed as a function of the observations (r, s) in addition to the base rate a.

$$\begin{cases} \alpha &= r + Wa \\ \beta &= s + W(1-a) \,, \end{cases} \tag{3.3}$$

so that an alternative representation of the Beta pdf is:

$$\begin{aligned} &\text{Beta}(p \,|\, r, s, a) = \tfrac{\Gamma(r+s+W)}{\Gamma(r+Wa)\Gamma(s+W(1-a))} p^{(r+Wa-1)}(1-p)^{(s+W(1-a)-1)} \,, \\ &\text{where } 0 \le p \le 1,\ (r+Wa) > 0,\ (s+W(1-a)) > 0 \,, , \end{aligned} \tag{3.4}$$

with the restriction that the probability variable $p \neq 0$ if $(r + Wa) < 1$, and $p \neq 1$ if $(s + W(1-a)) < 1$.

The non-informative prior weight denoted by W is normally set to $W = 2$ which ensures that the prior (i.e. when $r = s = 0$) Beta pdf with default base rate $a = 0.5$ is a uniform pdf.

The probability expectation value of the Beta pdf is defined by Eq.(3.5) below:

$$\text{E}(\text{Beta}(p \,|\, \alpha, \beta)) = \alpha/(\alpha+\beta) = \frac{r + Wa}{r + s + W} \,. \tag{3.5}$$

3.1.3 Binomial Opinion to Beta Mapping

The mapping from the parameters of a binomial opinion $\omega_x = (b, d, u, a)$ to the parameters of a Beta pdf $\mathrm{Beta}(p \,|\, r, s, a)$ is defined by:

Definition 6 (Binomial Opinion-Beta Mapping) .
Let $\omega_x = (b, d, u, a)$ *be a binomial opinion, and let* $\mathrm{Beta}(p \mid r, s, a)$ *be a Beta pdf, both over the same proposition* x*, or in other words over the binary state space* $\{x, \overline{x}\}$*. The opinions* ω_x *and* $\mathrm{Beta}(p \mid r, s, a)$ *are equivalent through the following mapping:*

$$\begin{cases} b = \frac{r}{W+r+s} \\ d = \frac{s}{W+r+s} \\ u = \frac{W}{W+r+s} \end{cases} \Leftrightarrow \left(\begin{array}{ll} \underline{\text{For } u \neq 0:} & \underline{\text{For } u = 0:} \\ \begin{cases} r = \frac{Wb}{u} \\ s = \frac{Wd}{u} \\ 1 = b + d + u \end{cases} & \begin{cases} r = b\,\infty \\ s = d\,\infty \\ 1 = b + d \end{cases} \end{array} \right) \quad (3.6)$$

A generalisation of this mapping is provided in Def.8 below. The default non-informative prior weight W is normally defined as $W = 2$ because it produces a uniform Beta pdf in case of default base rate $a = 1/2$. It can be seen from Eq.(3.6) that the vacuous binomial opinion $\omega_x = (0, 0, 1, \frac{1}{2})$ is equivalent to the uniform pdf $\mathrm{Beta}(p \,|\, 1, 1)$.

The example $\mathrm{Beta}(p \,|\, 2.53, 4.13)$ is illustrated in Fig.3.2. Through Eq.(3.6) this Beta pdf is equivalent to the example opinion $\omega_x = (0.2,\ 0.5,\ 0.3,\ 0.6)$ from Fig.3.1.

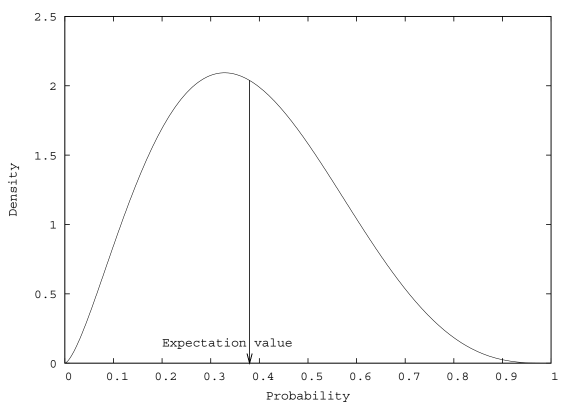


Figure 3.2: Probability density function $\mathrm{Beta}(p \,|\, 2.53, 4.13) \;\equiv\; \omega_x = (0.2,\ 0.5,\ 0.3,\ 0.6)$

In the example of Fig.3.2 where $\alpha = 2.53$ and $\beta = 4.13$ the probability expectation is $\mathrm{E}(\mathrm{Beta}(p \,|\, \alpha,\beta)) = 2.53/6.66 = 0.38$ which is indicated with the vertical arrow. This value is of course equal to that of Fig.3.1 because the Beta pdf is equivalent to the opinion through Eq.(3.6). The equivalence between binomial opinions and Beta pds is very powerful because subjective logic operators then can be applied to density functions and vice versa, and also because binomial opinions can be determined through statistical observations. Multinomial opinions described next are a generalisation of binomial opinions in the same way as Dirichlet pdfs are a generalisaiton of Beta pdfs.

3.2 Multinomial Opinions

3.2.1 The Multinomial Opinion Representation

An opinion on a frame X larger than binary where the set of focal elements is restricted to class-1 elements in addition to X itself is called a multinomial opinion. The special characteristic if this opinion class is thus that possible focal elements in $\mathcal{R}(X)$ are always singletons of X which by by definition are never overlapping. The frame X can have uncertainty mass assigned to it, but is not considered as a focal element. In case $u_X \neq 0$ it is called a UMO (Uncertain Multinomial Opinion), and in case $u_X = 0$ it is called a DMO (Dogmatic Multinomial Opinion).

In case of multinomial opinions the belief vector $\vec{b}_X$ and the base rate vector $\vec{a}_X$ both have k parameters each. The uncertainty parameter u_X is a simple scalar. A multinomial opinion thus contains $(2k+1)$ parameters. However, given Eq.(2.4) and Eq.(2.5), multinomial opinions only have $(2k-1)$ degrees of freedom. It is interesting to note that for binary state spaces there is no difference between hyper opinions and multinomial opinions, because uncertain binomial opinions are always 3-dimensional.

Visualising multinomial opinions is not trivial. The largest opinions that can be easily visualised are trinomial, in which case it can be presented as a point inside a tetrahedron, as shown in Fig.3.3.

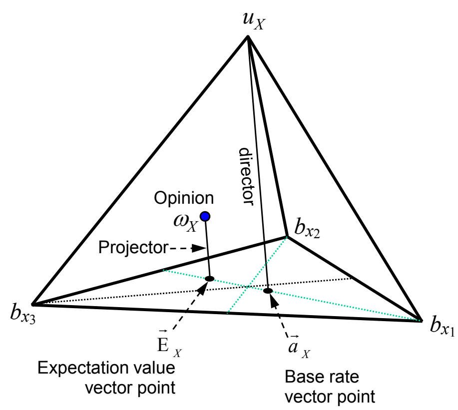


Figure 3.3: Opinion tetrahedron with example opinion

In Fig.3.3, the vertical elevation of the opinion point inside the tetrahedron represents the uncertainty mass. The distances from each of the three triangular side planes to the opinion point represents the respective belief mass values. The base rate vector $\vec{a}_X$ is indicated as a point on the base plane. The line that joins the tetrahedron summit and the base rate vector point represents the director. The probability expectation vector point is geometrically determined by drawing a projection from the opinion point parallel to the director onto the base plane.

In general, the triangle and tetrahedron belong to the *simplex* family of geometrical shapes. Multinomial opinions on frames of cardinality k can in general be represented as a point in a simplex of dimension $(k + 1)$. For example, binomial opinions can be represented inside a triangle which is a 3D simplex, and trinomial opinions can be represented inside a tetrahedron which is a 4D The 2D aspect of paper and computer screens makes visualisation of larger multinomial opinions impractical. Opinions with dimensions larger than trinomial do not lend themselves to traditional visualisation.

The probability projection of multinomial opinions is relatively simple to calculate compared to general opinions because no focal elements are overlapping. The expression for the probability expectation value of multinomial opinions is therefore a special case of the general expression of Def.3.14. The probability projection of multinomial opinions is defined by Eq.(3.7) below.

$$\vec{E}_X(x_i) = \vec{b}_X(x_i) \ + \ \vec{a}_X(x_i)\, u_X \ , \qquad \forall\ x_i \in X \ . \tag{3.7}$$

It can be noted that the probability projection of multinomial opinions expressed by Eq.(3.7) is a generalisation of the probability projection of binomial opinions expressed by Eq.(3.1).

3.2.2 The Dirichlet Multinomial Model

A UM opinion is equivalent to a Dirichlet pdf over X according to a specific mapping. For self-containment, we briefly outline the Dirichlet multinomial model below, and refer to [3] for more details.

The general multinomial probability density over a frame of cardinality k is described by the k-dimensional Dirichlet density function, where the special case of a probability density over a binary frame (i.e. where $k = 2$) is described by the Beta density function. In general, the Dirichlet density function can be used to represent probability density of a k-component stochastic probability vector variable $\vec{p}(x_i)$, $i = 1 \dots k$ with sample space $[0, 1]^k$ where $\sum_{i=1}^{k} vecp(x_i) = 1$. Because of this additivity requirement the Dirichlet density function has only $k-1$ degrees of freedom. This means that the knowledge of $k - 1$ probability variables and their density determines the last probability variable and its density.

The Dirichlet density function takes as argument a sequence of observations of the k possible outcomes represented as k positive real parameters $\vec{\alpha}(x_i)$, $i = 1 \dots k$, each corresponding to one of the possible outcomes.

Definition 7 Dirichlet Density Function

Let X be a frame consisting of k mutually disjoint elements. Let $\vec{\alpha}$ represent the evidence vector over the elements of X. In order to have a compact notation we define the vector $\vec{p} = \{\vec{p}(x_i) \mid 1 \le i \le k\}$ to denote the k-component random probability variable, and the vector $\vec{\alpha} = \{\vec{\alpha}(x_i) \mid 1 \le i \le k\}$ to denote the k-component random input argument vector $[\vec{\alpha}(x_i)]_{i=1}^{k}$. Then the multinomial Dirichlet density function over X, denoted as Dirichlet$(\vec{p} \mid \vec{\alpha})$, *is expressed as:*

$$\text{Dirichlet}(\vec{p} \mid \vec{\alpha}) = \frac{\Gamma\left(\sum_{i=1}^{k} \vec{\alpha}(x_i)\right)}{\prod_{i=1}^{k} \Gamma(\vec{\alpha}(x_i))} \prod_{i=1}^{k} p(x_i)^{(\vec{\alpha}(x_i)-1)} \tag{3.8}$$

where $\vec{\alpha}(x_1), \dots \vec{\alpha}(x_k) \ge 0$.

The vector $\vec{\alpha}$ represents the *a priori* as well as the observation evidence. The non-informative prior weight is expressed as a constant W, and this weight is distributed over all the possible outcomes as a function of the base rate. It is normally assumed that $W = 2$.

The singleton elements in a frame of cardinality k can have a base rate different from the default value $1/k$. It is thereby possible to define a base rate as a vector $\vec{a}$ with arbitrary distribution over the k mutually disjoint elements x_i, as long as the simple additivity requirement expressed as $\sum_{x_i \in X} \vec{a}(x_i) = 1$ is satisfied. The total evidence $\alpha(x_i)$ for each element $x_i \in X$ can then be expressed as:

$$\vec{\alpha}(x_i) = \vec{r}(x_i) + W\,\vec{a}(x_i)\,, \quad \text{where} \left\{ \begin{array}{l} \vec{r}(x_i) \ge 0 \\ \vec{a}(x_i) \in [0, 1] \\ \sum_{i=1}^{k} \vec{a}(x_i) = 1 \\ W \ge 2 \end{array} \right\} \quad \forall x_i \in X\,. \tag{3.9}$$

The Dirichlet density over a set of k possible states x_i can thus be represented as a function of the base rate vector $\vec{a}$, the non-informative prior weight W and the observation evidence $\vec{r}$.

The notation of Eq.(3.9) is useful, because it allows the determination of the probability densities over frames where each element can have an arbitrary base rate. Given the Dirichlet density function of Eq.(3.8), the probability expectation of any of the k random probability variables can now be written as:

$$\vec{\mathrm{E}}_X(x_i \mid \vec{r}, \vec{a}) = \frac{\vec{r}(x_i) + W\vec{a}(x_i)}{W + \sum_{x_j \in X} \vec{r}(x_j)} \qquad \forall x_i \in X\,, \tag{3.10}$$

which represents a generalisation of the probability expectation of the Beta pdf expressed by Eq.(3.5).

It is normally assumed that the *a priori* probability density in case of a binary frame $X = \{x, \overline{x}\}$ is uniform. This requires that $\vec{\alpha}(x) = \vec{\alpha}(\overline{x}) = 1$, which in turn dictates that $W = 2$. Assuming an *a priori* uniform density over frames other than binary will require a different value for W. The non-informative prior weight W will always be equal to the cardinality of the frame over which a uniform density is assumed.

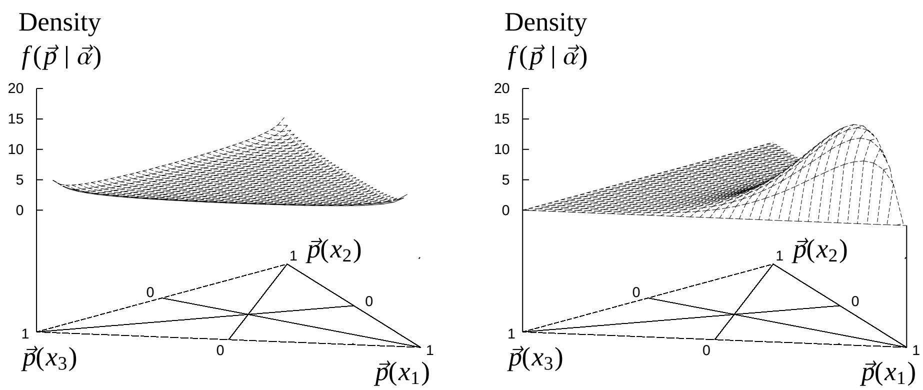
Selecting $W > 2$ will result in new observations having relatively less influence over the Dirichlet density function. This could be meaningful e.g. as a representation of specific *a priori* information provided by a domain expert. It can be noted that it would be unnatural to require a uniform density over arbitrary large frames because it would make the sensitivity to new evidence arbitrarily small.

For example, requiring a uniform *a priori* density over a frame of cardinality 100, would force $W = 100$. In case an event of interest has been observed 100 times, and no other event has been observed, the derived probability expectation of the event of interest will still only be about $\frac{1}{2}$, which would be rather counterintuitive. In contrast, when a uniform density is assumed in the binary case, and the same evidence is analysed, the derived probability expectation of the event of interest would be close to 1, as intuition would dictate.

3.2.3 Visualising Dirichlet Probability Density Functions

Visualising Dirichlet density functions is challenging because it is a density function over $k-1$ dimensions, where k is the frame cardinality. For this reason, Dirichlet density functions over ternary frames are the largest that can be practically visualised.

Let us consider the example of an urn containing balls of the three different markings: x_1, x_2 and x_3, meaning that the urn can contain multiple balls marked x_1, x_2 and x_3 respectively. Because of the three different markings, the cardinality of this frame is 3 (i.e. $k = 3$). Let us first assume that no other information than the cardinality is available, meaning that the number and relative proportion of balls marked x_1, x_2 and x_3 are unknown, and that the default base rate for any of the markings is $a = 1/k = \frac{1}{3}$. Initially, before any balls have been drawn we have $\vec{r}(x_1) = \vec{r}(x_2) = \vec{r}(x_3) = 0$. Then Eq.(3.10) dictates that the expected *a priori* probability of picking a ball of any specific marking is the default base rate probability $a = \frac{1}{3}$. The non-informative *a priori* Dirichlet density function is illustrated in Fig.3.4.a.



(a) Non-informative prior Dirichlet density

(b) *A posteriori* Dirichlet density

Figure 3.4: Prior and posterior Dirichlet density functions

Let us now assume that an observer has picked (with return) 6 balls market x_1, 1 ball marked x_2 and 1 ball marked x_3, i.e. $\vec{r}(x_1) = 6$, $\vec{r}(x_2) = 1$, $\vec{r}(x_3) = 1$, then the *a posteriori* expected probability of picking a ball marked x_1 can be computed as $\vec{\mathrm{E}}_X(x_1) = \frac{2}{3}$. The *a posteriori* Dirichlet density function is illustrated in Fig.3.4.b.

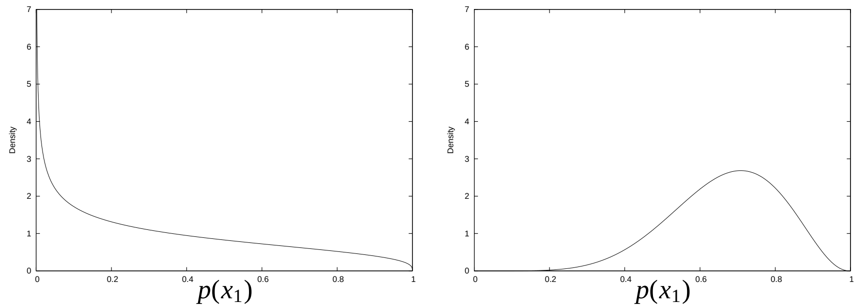
3.2.4 Coarsening Example: From Ternary to Binary

We reuse the example of Sec.3.2.3 with the urn containing balls marked x_1, x_2 and x_3, but this time we consider a binary partition of the markings into x_1 and $\overline{x}_1 = \{x_2, x_3\}$. The base rate of picking x_1 is set to the relative atomicity of x_1, expressed as $\vec{a}(x_1) = \frac{1}{3}$. Similarly, the base rate of picking $\overline{x}_1$ is $\vec{a}(\overline{x}_1) = \vec{a}(x_2) + \vec{a}(x_3) = \frac{2}{3}$.

Let us again assume that an observer has picked (with return) 6 balls marked x_1, and 2 balls marked $\overline{x}_1$, i.e. marked x_2 or x_3. This translates into the evidence vector $\vec{r}(x_1) = 6$, $\vec{r}(\overline{x}_1) = 2$.

Since the frame has been reduced to binary, the Dirichlet density function is reduced to a Beta density function which is simple to visualise. The *a priori* and *a posteriori* density functions are illustrated in Fig.3.5.

Computing the *a posteriori* expected probability of picking ball marked x_1 with Eq.(3.10) produces $\vec{\mathrm{E}}_X(x_1) = \frac{2}{3}$, which is the same as before the coarsening, as illustrated in Sec.3.2.3. This shows that the coarsening does not



(a) Non-informative *a priori* Beta probability density

(b) After 6 balls marked x_1, and 2 balls marked x_2 or x_3

Figure 3.5: Prior and posterior Beta probability density

influence the probability expectation value of specific events.

3.2.5 Multinomial Opinion - Dirichlet Mapping

Dirichlet density functions translate observation evidence directly into probability densities over a state space. The representation of the observation evidence, together with the base rate, can be used to determine opinions. In other words is possible to define a bijective mapping between Dirichlet pdfs and multinomial opinions over frame X.

Let $X = \{x_i \mid i = 1, \cdots, k\}$ be a frame. Let $\omega_X = (\vec{b}, u, \vec{a})$ be an opinion on X, and let Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$ be a Dirichlet pdf over X.

For the bijective mapping between ω_X and Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$, we require equality between the probability expectation value $\vec{\mathrm{E}}_X(x_i)$ derived from ω_X, and that derived from Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$. This constraint is expressed as:

For all $x_i \in X$:

$$\vec{\mathrm{E}}(\omega_X) = \vec{\mathrm{E}}(\text{Dirichlet}(\vec{p} \mid \vec{r}, \vec{a})) \quad (3.11)$$

$$\Updownarrow$$

$$\vec{b}(x_i) + \vec{a}(x_i)u = \frac{\vec{r}(x_i)}{W + \sum_{i=1}^{k} \vec{r}(x_i)} + \frac{W\vec{a}(x_i)}{W + \sum_{i=1}^{k} \vec{r}(x_i)} \quad (3.12)$$

We also require that each belief mass $\vec{b}(x_i)$ be an increasing function of of the evidence $\vec{r}(x_i)$, and that u be a decreasing function of $\sum_{i=1}^{k} \vec{r}(x_i)$. In other words, the more evidence in favour of a particular outcome x_i, the greater the belief mass on that outcome. Furthermore, the more total evidence available, the less uncertain the opinion. As already mentioned it is normally assumed that $W = 2$.

In case $u \longrightarrow 0$, then $\sum_{i=1}^{k} \vec{b}(x_i) \longrightarrow 1$, and $\sum_{i=1}^{k} \vec{r}(x_i) \longrightarrow \infty$, meaning that at least some, but not necessarily all, of the evidence parameters $\vec{r}(x_i)$ are infinite.

These intuitive requirements together with Eq.(3.12) provide the basis for the following bijective mapping:

Definition 8 Multinomial Opinion-Dirichlet Mapping
Let $\omega_X = (\vec{b}, u, \vec{a})$ be a multinomial opinion, and let Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$ *be a Dirichlet pdf, both over the same frame X of cardinality k. The multinomial opinion ω_X and the pdf* Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$ *are equivalent through the following*

mapping:

$$\forall x_i \in X \quad \begin{cases} \vec{b}(x_i) = \frac{\vec{r}(x_i)}{W + \sum_{i=1}^{k} \vec{r}(x_i)} \\ u = \frac{W}{W + \sum_{i=1}^{k} \vec{r}(x_i)} \end{cases} \Leftrightarrow \left(\begin{array}{ll} \underline{\text{For } u \neq 0:} & \underline{\text{For } u = 0:} \\ \begin{cases} \vec{r}(x_i) = \frac{W\vec{b}(x_i)}{u} \\ 1 = u + \sum_{i=1}^{k} \vec{b}(x_i) \end{cases} & \begin{cases} \vec{r}(x_i) = \vec{b}(x_i)\,\infty \\ 1 = \sum_{i=1}^{k} \vec{b}(x_i) \end{cases} \end{array} \right) \tag{3.13}$$

The mapping of Def.8 is a generalisation of the binomial mapping from Def.6. The interpretation of Beta and Dirichlet pdfs is well established in the statistics community so that the mapping of Def.8 creates a direct mathematical and interpretational equivalence between Dirichlet pdfs and opinions when both are expressed over the same frame X.

One the one hand, statistical tools and methods such as collecting statistical observation evidence can now be applied to opinions. On the other hand, the operators of subjective logic such as conditional deduction and abduction can be applied to statistical evidence.

The introduction of hyper opinions which will be described below, also creates an equivalence to probability density functions over powersets which have received relatively little attention in the literature. The rest of the paper is concerned with the interpretation of hyper opinions as equivalent representations of this class of probability density functions.

3.3 Hyper Opinions

3.3.1 The Hyper Opinion Representation

An opinion on a frame X of cardinality $k > 2$ where any element $x \in \mathcal{R}(X)$ can be a focal element is called a hyper opinion. The special characteristic if this opinion class is thus that possible focal elements $x \in \mathcal{R}(X)$ can be overlapping subsets of the frame X. The frame X itself can have uncertainty mass assigned to it, but is not considered as a focal element. Definition 9 below not only defines hyper opinions, but also represents a general definition of subjective of opinions. In case $u_X \neq 0$ it is called a UH opinion (uncertain hyper opinion), and in case $u_X = 0$ it is called a DH opinion (dogmatic hyper opinion).

Definition 9 Hyper Opinion
Assume X be to a frame where $\mathcal{R}(X)$ denotes its reduced powerset. Let $\vec{b}_X$ be a belief vector over the elements of $\mathcal{R}(X)$, let u_X be the complementary uncertainty mass, and let $\vec{a}$ be a base rate vector over the frame X, all seen from the viewpoint of the opinion owner A. The composite function $\omega_X^A = (\vec{b}_X, u_X, \vec{a}_X)$ is then A's hyper opinion over X.

The belief vector $\vec{b}_X$ has $(2^k - 2)$ parameters, whereas the base rate vector $\vec{a}_X$ only has k parameters. The uncertainty parameter u_X is a simple scalar. A hyper opinion thus contains $(2^k + k - 1)$ parameters. However, given Eq.(2.4) and Eq.(2.5), hyper opinions only have $(2^k + k - 3)$ degrees of freedom.

Hyper opinions represent the most general class of opinions. It is challenging to design meaningful visualisations of hyper opinions because belief masses are distributed over the reduced powerset with partly overlapping elements.

An element of $x_i \in \mathcal{R}(X)$ is called a *focal element* when its belief mass is non-zero, i.e. when $\vec{b}_X(x_i) \neq 0$.

The integration of the base rates in opinions allows the probability projection to be independent from the internal structure of the frame. The probability expectation of hyper opinions is a vector expressed as a function of the belief vector, the uncertainty mass and the base rate vector.

Definition 10 (Probability Projection of Hyper Opinions) *Assume X to be a frame of cardinality k where $\mathcal{R}(X)$ is its reduced powerset of cardinality $\kappa = (2^k - 2)$. Let $\omega_X = (\vec{b}_X,\ u_X,\ \vec{a}_X)$ be a hyper opinion on X. The probability projection of hyper opinions is defined by the vector $\vec{\mathrm{E}}_X$ from $\mathcal{R}(X)$ to $[0,1]^\kappa$ expressed as:*

$$\vec{\mathrm{E}}_X(x_i) = \sum_{x_j \in \mathcal{R}(X)} \vec{a}_X(x_i/x_j)\, \vec{b}_X(x_j) \ + \ \vec{a}_X(x_i)\, u_X\ , \quad \forall\ x_i \in \mathcal{R}(X)\ . \tag{3.14}$$

For $x \in X$ it can be shown that the probability projection $\vec{\mathrm{E}}_X$ satisfies the probability additivity principle:

$$\vec{\mathrm{E}}_X(\emptyset) = 0 \quad \text{and} \quad \sum_{x \in X} \vec{\mathrm{E}}_X(x) = 1\ . \tag{3.15}$$

However, $x \in \mathcal{R}(X)$, the sum of probability projections is in general super-additive, formally expressed as:

$$\sum_{x \in \mathcal{R}(X)} \vec{\mathrm{E}}_X(x) \geq 1\ . \tag{3.16}$$

3.3.2 The Hyper Dirichlet Model over the Reduced Powerset

Probability density over the reduced powerset $\mathcal{R}(X)$ can be described by the Dirichlet density function similarly to the way that it represents probability density over the frame X. Because any subset of X can be a focal element for a hyper opinion, evidence parameters in the Dirichlet pdf apply to the same subsets of X.

Let $k = |X|$ be the cardinality of X so that $\kappa = |\mathcal{R}(X)| = (2^k - 2)$ is the cardinality of $\mathcal{R}(X)$. In case of hyper opinions, the Dirichlet density function represents probability density on a κ-dimensional stochastic probability variable $p(x_i)$, $i = 1 \dots \kappa$ associated with the reduced powerset $\mathcal{R}(X)$.

The input arguments are now a sequence of observations of the κ possible elements $x_i \in \mathcal{R}(X)$ represented as κ positive real parameters $\vec{\alpha}(x_i)$, $i = 1 \dots \kappa$, each corresponding to one of the possible observations.

Definition 11 (Dirichlet Density Function over $\mathcal{R}(X)$)
Let X be a frame consisting of k mutually disjoint elements, where the reduced powerset $\mathcal{R}(X)$ has cardinality $\kappa = (2^k - 2)$. Let $\vec{\alpha}$ represent the evidence vector over the elements of $\mathcal{R}(X)$. In order to have a compact notation we define the vector $\vec{p} = \{\vec{p}(x_i) \mid 1 \leq i \leq \kappa\}$ to denote the κ-component random probability variable, and the vector $\vec{\alpha} = \{\vec{\alpha}(x_i) \mid 1 \leq i \leq \kappa\}$ to denote the κ-component random input argument vector $[\vec{\alpha}(x_i)]_{i=1}^{\kappa}$. Then the Dirichlet density function over $\mathcal{R}(X)$, denoted as power-$D(\vec{p} \mid \vec{\alpha})$, is expressed as:

$$D(\vec{p} \mid \vec{\alpha}) = \frac{\Gamma\left(\sum_{i=1}^{\kappa} \vec{\alpha}(x_i)\right)}{\prod_{i=1}^{\kappa} \Gamma(\vec{\alpha}(x_i))} \prod_{i=1}^{\kappa} p(x_i)^{(\vec{\alpha}(x_i)-1)} \tag{3.17}$$

where $\vec{\alpha}(x_1), \dots \vec{\alpha}(x_\kappa) \geq 0$.

The vector $\vec{\alpha}$ represents the *a priori* as well as the observation evidence. The non-informative prior weight is expressed as a constant $W = 2$, and this weight is distributed over all the possible outcomes as a function of the base rate.

Since the elements of $\mathcal{R}(X)$ can contain multiple singletons from X, an element of $\mathcal{R}(X)$ has a base rate equal to the sum of base rates of the singletons it contains. This results in a super-additive base rate vector $\vec{a}_X$ over $\mathcal{R}(X)$. The total evidence $\alpha(x_i)$ for each element $x_i \in \mathcal{R}(X)$ can then be expressed as:

$$\vec{\alpha}(x_i) = \vec{r}(x_i) + W\, \vec{a}_X(x_i)\,, \quad \text{where} \left\{ \begin{array}{l} \vec{r}(x_i) \geq 0 \\ \\ \vec{a}_X(x_i) = \sum\limits_{\substack{x_j \subseteq x_i \\ x_j \in X}} \vec{a}(x_j) \\ \\ W \geq 2 \end{array} \right\} \quad \forall x_i \in \mathcal{R}(X) \tag{3.18}$$

The Dirichlet density over a set of κ possible states $x_i \in \mathcal{R}(X)$ can thus be represented as a function of the super additive base rate vector $\vec{a}_X$ (for $x_i \in \mathcal{R}(X)$), the non-informative prior weight W and the observation evidence $\vec{r}$.

Because subsets of X can be overlapping, the probability expectation value of any subset $x_i \subset X$ (equivalent to an element $x_i \in \mathcal{R}(X)$) is not only a function of the direct probability density on x_i, but also of the probability density of all other subsets that are overlapping with x_i (partially or totally). More formally, the probability expectation of a subset $x_i \subset X$ results from the probability density of each $x_j \subset X$ where $x_i \cap x_j \neq \emptyset$.

Given the Dirichlet density function of Eq.(3.17), the probability expectation of any of the κ random probability variables can now be written as:

$$\vec{\mathrm{E}}_X(x_i \mid \vec{r}, \vec{a}) = \frac{\sum\limits_{x_j \in \mathcal{R}(X)} \vec{a}_X(x_i/x_j)\vec{r}(x_j) + W\vec{a}_X(x_i)}{W + \sum\limits_{x_j \in \mathcal{R}(X)} \vec{r}(x_j)} \qquad \forall x_i \in \mathcal{R}(X)\ . \tag{3.19}$$

The probability expectation vector of Eq.(3.19) is a generalisation of the probability expectation vector of Eq.(3.10).

3.3.3 The Hyper Opinion - Hyper Dirichlet Mapping

A hyper opinion is equivalent to a Dirichlet pdf over the reduced powerset $\mathcal{R}(X)$ according a mapping which simply is an extension of the mapping used for multinomial opinions defined in Def.8.

Definition 12 Hyper Opinion to Dirichlet Mapping
Let $\omega_X = (\vec{b}, u, \vec{a})$ *be a hyper opinion on* X *of cardinality* k*, and let* Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$ *be a Dirichlet pdf over* $\mathcal{R}(X)$ *of cardinality* $\kappa = (2^k - 2)$*. The hyper opinion* ω_X *and the pdf* Dirichlet$(\vec{p} \mid \vec{r}, \vec{a})$ *are equivalent through the following mapping:*

$$\forall x_i \in \mathcal{R}(X) \quad \begin{cases} \vec{b}(x_i) = \frac{\vec{r}(x_i)}{W + \sum_{i=1}^{\kappa} \vec{r}(x_i)} \\ u = \frac{W}{W + \sum_{i=1}^{\kappa} \vec{r}(x_i)} \end{cases} \Leftrightarrow \left(\begin{array}{c|c} \text{For } u \neq 0\text{:} & \text{For } u = 0\text{:} \\ \begin{cases} \vec{r}(x_i) = \frac{W\vec{b}(x_i)}{u} \\ 1 = u + \sum\limits_{i=1}^{\kappa} \vec{b}(x_i) \end{cases} & \begin{cases} \vec{r}(x_i) = \vec{b}(x_i)\,\infty \\ 1 = \sum\limits_{i=1}^{\kappa} \vec{b}(x_i) \end{cases} \end{array} \right) \tag{3.20}$$

The mapping of Def.12 is a generalisation of the multinomial mapping from Def.8. However, the interpretation of Dirichlet pdfs over partly overlapping elements of a frame is not well studied. A Dirichlet pdf over $\mathcal{R}(X)$ when projected over X is not a Dirichlet pdf in general. Only a few degenerate cases become Dirichlet pdfs through this projection, such as the non-informative prior Dirichlet where $\vec{r}$ is the zero vector which corresponds to a vacuous opinion with $u = 1$, or the case where all the focal elements are pairwise disjoint. While it is challenging to work with the mathematical representation of such non-conventional Dirichlet pdfs, we now have the advantage that there exists an equivalent representation in the form of hyper opinions.

It would not be meaningful to try to visualise the Dirichlet pdf over $\mathcal{R}(X)$ on $\mathcal{R}(X)$ itself because it would fail to visualise the important fact that some focal elements are overlapping in X. A visualisation of the Dirichlet pdf over $\mathcal{R}(X)$ should therefore be done on X. This can be done by integrating the evidence parameters of the Dirichlet pdf over $\mathcal{R}(X)$ in a pdf over X. In other words, the contribution from the overlapping random variables must be combined with the Dirichlet pdf over X. A method for doing exactly this, defined by Hankin [4], produces a *hyper Dirihclet pdf* which is a generalisation of the standard Dirichlet model. In addition to the factors consisting of the probability product of the random variables, it requires a normalisation factor $B(\vec{\alpha})$ that can be computed numerically. Hankin also provides a software package for producing visualisations of hyper Dirichlet pdfs over ternary frames.

The mathematical expression of a hyper Dirichlet pdf, denoted as hyper-D$(\vec{p}|\vec{\alpha})$ is given by Eq.(3.21) below.

$$\text{hyper-D}(\vec{p}|\vec{\alpha}) = B(\vec{\alpha}) \left(\prod_{i=1}^{k} p(x_i)^{(\vec{\alpha}(x_i)-1)} \prod_{j=(k+1)}^{\kappa} p(x_j)^{\vec{r}(x_j)} \right) \quad \text{where}$$

$$B(\vec{\alpha}) = \int\limits_{\substack{p(x)\geq 0 \\ \sum_{j=(k+1)}^{\kappa} p(x_j) \leq 1}} \left(\prod_{i=1}^{k} p(x_i)^{(\vec{\alpha}(x_i)-1)} \prod_{j=(k+1)}^{\kappa} p(x_j)^{\vec{r}(x_j)} \right) d(p(x_1), \ldots p(x_\kappa)) \tag{3.21}$$

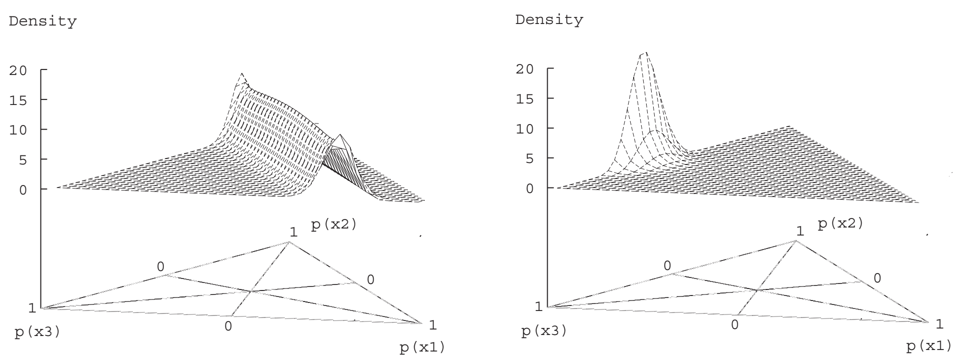
The ability to represent statistical observations in terms of hyper Dirichlet pdfs can be useful in many practical situations. We will here consider the example of a genetical engineering process where eggs of 3 different mutations are being produced. The mutations are denoted by $x_1, x_2 and x_3$ respectively so that the frame can be defined as

$X = \{x_1, x_2 x_3\}$. The specific mutation of each egg can not be controlled by the process, so a sensor is being used to determine the mutation of each egg. Let us assume that the sensor is not always able to determine the mutation exactly, and that it sometimes can only exclude one out of the three possibilities. What is observed by the sensors is therefore elements of the reduced powerset $\mathcal{R}(X)$. We consider two separate scenarios of 100 observations. In scenario A, mutation x_3 has been observed 20 times, and mutation x_1 or x_2 (i.e. the element $\{x_1, x_2\}$) has been observed 80 times. In scenario B, mutation x_2 has been observed 20 times, the mutations x_1 or x_3 (i.e. the element $\{x_1, x_3\}$) have been observed 40 times, and the mutations x_2 or x_3 (i.e. the element $\{x_2, x_3\}$) have also been observed 40 times. Table 3.2 summarises the two scenarios. The base rate is set to the default value of 1/3 for each mutation..

	Scenario A						Scenario B					
Mutation:	x_1	x_2	x_3	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	x_1	x_2	x_3	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
Counts:	0	0	20	80	0	0	0	20	0	0	40	40

Table 3.2: Number of observations per mutation category

Because the frame X is ternary it is possible to visualise the corresponding hyper Dirichlet pdfs, as shown in Fig.3.6



(a) Hyper Dirichlet pdf of scenario A (b) Hyper Dirichlet pdf of scenario B

Figure 3.6: Example hyper Dirichlet probability density functions

Readers who are familiar with the typical shapes of Dirichlet pdfs will immediately notice that the plots of Fig.3.6 are clearly not Dirichlet. The hyper Dirichlet [4] represents a generalisation of the classic Dirichlet and provides a nice interpretation of hyper opinions.

3.4 Alternative Opinion Representations

3.4.1 Probabilistic Notation of Opinions

A disadvantage of the belief and evidence notations described in Sec.3 is that they do not explicitly express probability expectation values of the elements in the frame. The classical probabilistic notation has the advantage that it is used in all areas of science and that people are familiar with it. The probability expectation of an opinion can easily be derived with Eq.(3.14), but this still represents a mental barrier to a direct intuitive interpretation of opinions. An intuitive representation of multinomial opinions could therefore be to represent the probability expectation value directly, together with the degree of uncertainty and the base rate function. This will be called the *probabilistic notation* of opinions:

Definition 13 Probabilistic Notation of Opinions
Let X be a frame and let ω_X^{bn} be an opinion on X in belief notation. Let $\vec{\mathrm{E}}$ be a multinomial probability expectation

function on X defined according to Def.10, let $\vec{a}$ be a multinomial base rate function on X defined according to Def.2, and let $c = 1 - u$ be the certainty function on X, where u is defined according to Def.(5). The probabilistic notation of opinions can then be expressed as the ordered tuple $\omega_X^{\text{pn}} = (\vec{\text{E}}, c, \vec{a})$.

In case $c = 1$, then $\vec{\text{E}}$ is a traditional discrete probability distribution without uncertainty. In case $c = 0$, then $\vec{\text{E}} = \vec{a}$, and no evidence has been received, so the probability distribution $\vec{\text{E}}$ is totally uncertain.

The equivalence between the belief notation and the probabilistic notation of opinions is defined below.

Theorem 1 Probabilistic Notation Equivalence
Let $\omega_X^{\text{bn}} = (\vec{b}, u, \vec{a})$ be an opinion expressed in belief notation, and $\omega_X^{\text{pn}} = (\vec{\text{E}}, c, \vec{a})$ be an opinion expressed in probabilistic notation, both over the same frame X. The opinions ω_X^{bn} and ω_X^{pn} are equivalent when the following equivalent mappings hold:

$$\begin{cases} \vec{\text{E}}(x_i) & = \vec{b}(x_i) + \vec{a}(x_i)u \\ c & = 1 - u \end{cases} \Leftrightarrow \begin{cases} \vec{b}(x_i) & = \vec{\text{E}}(x_i) - \vec{a}(x_i)(1-c) \\ u & = 1 - c \end{cases} \tag{3.22}$$

Let the frame X have cardinality k. Then both the base rate vector $\vec{a}$ and the probability expectation vector $\vec{\text{E}}(x_i)$ have $k - 1$ degrees of freedom due to the additivity property of Eq.(2.5) and Eq.(3.15). With the addition of the independent certainty parameter c, the probabilistic notation of opinions has $2k - 1$ degrees of freedom, as do the belief notation and the evidence notation of opinions.

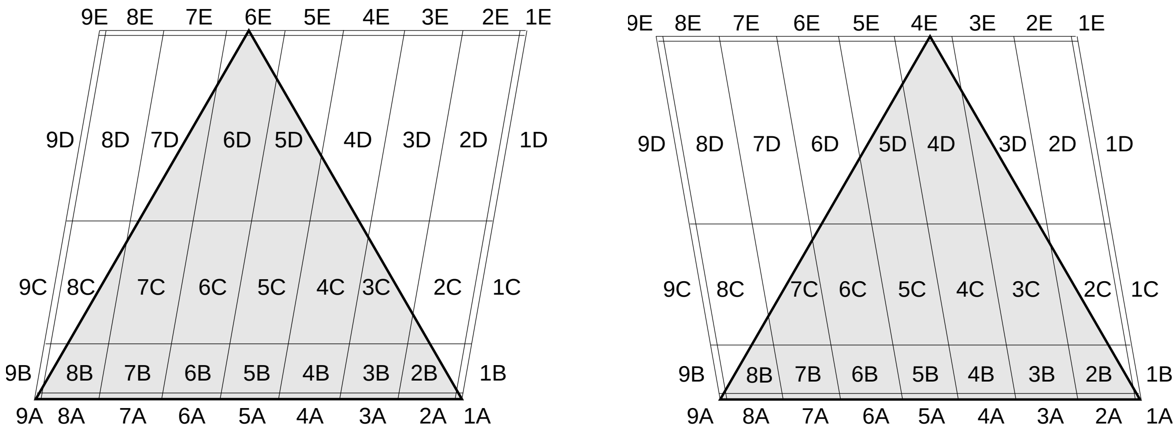
3.4.2 Fuzzy Category Representation

Human language provides various terms that are commonly used to express various types of likelihood and uncertainty. It is possible to express binomial opinions in terms of fuzzy verbal categories which can be specified according to the need of a particular application. An example set of fuzzy categories is provided in Table 3.3.

Likelihood Categories / **Certainty Categories**		Absolutely Not **9**	Very Unlikely **8**	Unlikely **7**	Somewhat Unlikely **6**	Chances about even **5**	Somewhat Likely **4**	Likely **3**	Very Lkely **2**	Absolutely **1**
Completely Uncertain	**E**	9E	8E	7E	6E	5E	4E	3E	2E	1E
Very Uncertain	**D**	9D	8D	7D	6D	5D	4D	3D	2D	1D
Uncertain	**C**	9C	8C	7C	6C	5C	4C	3C	2C	1C
Slightly Uncertain	**B**	9B	8B	7B	6B	5B	4B	3B	2B	1B
Completely Certain	**A**	9A	8A	7A	6A	5A	4A	3A	2A	1A

Table 3.3: Fuzzy Categories

These fuzzy verbal categories can be mapped to areas in the opinion triangle as illustrated in Fig.3.7. The mapping must be defined for combinations of ranges of expectation value and uncertainty. As a result, the mapping between a specific fuzzy category from Table 3.3 and specific geometric area in the opinion triangle depends on the base rate. Without specifying the exact underlying ranges, the visualization of Fig.3.7 indicates the ranges approximately. The edge ranges are deliberately made narrow in order to have categories for near dogmatic and vacuous beliefs, as well as beliefs that express expectation values near absolute 0 or 1. The number of likelihood categories, and certainty categories, as well as the exact ranges for each, must be determined according to the need of each application, and the fuzzy categories defined here must be seen as an example. Real-world categories would likely be similar to those found in Sherman Kent's *Words of Estimated Probability* [19]; based on the *Admiralty*



(a) Fuzzy categories with $a = \frac{1}{3}$

(b) Fuzzy categories with $a = \frac{2}{3}$

Figure 3.7: Mapping fuzzy categories to ranges of belief as a function of the base rate

Scale as used within the UK National Intelligence Model[2]; or could be based on empirical results obtained from psychological experimentation.

Fig.3.7 illustrates category-opinion mappings in the case of base rate $a = \frac{1}{3}$, and the case of base rate $a = \frac{2}{3}$. The mapping is determined by the overlap between category area and triangle region. Whenever a fuzzy category area overlaps, partly or completely, with the opinion triangle, that fuzzy category is a possible mapping.

Note that the fuzzy category areas overlap with different regions on the triangle depending on the base rate. For example, it can be seen that the category $7D$: "Unlikely and Very Uncertain" is possible in case $a = \frac{1}{3}$, but not in case $a = \frac{2}{3}$. This is because the expectation of a state x is defined as $\mathrm{E}(x) = b_x + a_x u_x$, so that when $a_x, u_x \longrightarrow 1$, then $\mathrm{E}(x) \longrightarrow 1$, meaning that the likelihood category "Unlikely" would be impossible.

Mapping from fuzzy categories to subjective opinions is also straight-forward. Geometrically, the process involves mapping the fuzzy adjectives to the corresponding center of the portion of the grid cell contained within the opinion triangle (see Fig.3.7). Naturally, some mappings will always be impossible for a given base rate, but these are logically inconsistent and should be excluded from selection.

It is interesting to notice that although a specific fuzzy category maps to different geometric areas in the opinion triangle depending on the base rate, it will always correspond to the same range of beta PDFs. It is simple to visualize ranges of binomial opinions with the opinion triangle, but it would not be easy to visualize ranges of beta PDFs. The mapping between binomial opinions and beta PDFs thereby provides a very powerful way of describing PDFs in terms of fuzzy categories, and vice versa.

3.5 Confusion Evidence in Opinions

Recall from Sec.2.3 that a confusion element is a subset $x_j \subset X$ that consists of two or more singletons, and that the confusion set denoted by $\mathcal{C}(X)$ is the set of all confusion elements from the frame. Belief mass that is assigned to a confusion element expresses cognitive confusion because belief mass of this type supports the truth of multiple singletons in X simultaneously, i.e. it does not discriminate between any of them. In case of binary frames there is no confusion element. In case of frames larger than binary there are always confusion elements, and every singleton $x_i \in X$ is member of multiple confusion elements. The confusion mass on a singleton $x_i \in X$ is defined as the sum of belief masses on the confusion elements of which x_i is a member. The total confusion mass is simply the sum of belief masses on all confusion elements in the frame. The formal definitions are provided below.

Definition 14 (Confusion Mass) *Let X be a frame where $\mathcal{R}(X)$ denotes its reduced powerset and $\mathcal{C}(X)$ denotes its confusion set. Let $x_i \in X$ denote a singleton in X and let $y_j \in \mathcal{C}(X)$ denote a confusion element in $\mathcal{C}(X)$. Confusion*

[2] http://www.policereform.gov.uk/implementation/natintellmodel.html

mass on a singleton x_i, denoted as $c(x_i)$, is a function defined as:

$$c(x_i) = \sum_{\substack{x_j \in \mathcal{C}(X) \\ x_i \in x_j}} \vec{b}(x_j) \ . \tag{3.23}$$

The total confusion mass in an opinion ω_X is defined as the sum of belief masses on confusion elements in $\mathcal{C}(X)$, formally expressed as:

$$c_{\mathrm{T}}(\omega_X) = \sum_{x_j \in \mathcal{C}(X)} \vec{b}(x_j). \tag{3.24}$$

The confusion masses on singletons can be grouped in a single vector denoted as $\vec{c}$.

An opinion ω_X is totally confusing when $c(\omega_X) = 1$ and partially confusing when $c(\omega_X) < 1$. An opinion has single confusion when only a single focal confusion element exists. Correspondingly an opinion has multi-confusion when multiple focal confusion elements exist. It can be verified that:

$$0 \leq \vec{c}(x_j) \leq c_{\mathrm{T}}(\omega_X) \leq 1 \quad \forall x_j \in \mathcal{C}(X) \ . \tag{3.25}$$

It is important to note that uncertainty and confusion are different concepts in subjective logic. Uncertainty reflects lack of evidence whereas confusion results from evidence that fails to discriminate between specific singletons. A totally uncertain opinion, by definition, does not contain any confusion. Hyper opinion contains confusion, but multinomial and binomial opinions do not contain confusion. The ability to express confusion is thus the main difference between hyper opinions and multinomial opinions. The confusion in a hyper opinion may increase or decrease over time as a function of new evidence. When assuming that evidence never decays uncertainty can only decrease over time because evidence is accumulated and never is lost. When assuming that evidence decays e.g. as a function of time then uncertainty can increase over time because it is equivalent with the loss of evidence.

Confusion can be indicated and visualised in various ways, e.g. on the opinion simplex. A tetrahedron is the largest simplex that can easily be visualised on paper or on a 2D computer display. Let us for example consider the ternary frame X which can be described as follows:

$$\begin{cases} \text{Frame:} & X = \{x_1, x_2, x_3\} \\ \text{Reduced powerset:} & \mathcal{R}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6\} \\ \text{Confusion set:} & \mathcal{C}(X) = \{x_4, x_5, x_6\} \end{cases} \quad \text{where} \quad \begin{cases} x_4 = (x_1 \cup x_2) \\ x_5 = (x_1 \cup x_3) \\ x_6 = (x_2 \cup x_3) \end{cases} \tag{3.26}$$

Let us further assume a hyper opinion ω_X with default base rates and with belief mass vector specified by $\vec{b}(x_4) = 0.6$ and $\vec{b}(x_5) = 0.4$. It can be noted that this opinion is dogmatic because $u_X = 0$ and totally multi-confusing because $c(\omega_X) = 1$ and because there are two focal confusion elements.

Applying Eq.(3.23) produces the confusion vector as:

$$\vec{c}(x_1) = 1.0, \quad \vec{\mathrm{E}}(x_2) = 0.6, \quad \vec{\mathrm{E}}(x_1) = 0.4 \ . \tag{3.27}$$

The probability expectation vector on X can be computed with Eq.(3.14) to produce:

$$\vec{\mathrm{E}}(x_1) = 0.5, \quad \vec{\mathrm{E}}(x_2) = 0.3, \quad \vec{\mathrm{E}}(x_1) = 0.2 \ . \tag{3.28}$$

Chapter 4

Operators of Subjective Logic

4.1 Generalising Probabilistic Logic as Subjective Logic

In case the argument opinions are equivalent to binary logic TRUE or FALSE, the result of any subjective logic operator is always equal to that of the corresponding propositional/binary logic operator. Similarly, when the argument opinions are equivalent to traditional probabilities, the result of any subjective logic operator is always equal to that of the corresponding probability operator.

In case the argument opinions contain degrees of uncertainty, the operators involving multiplication and division will produce derived opinions that always have correct expectation value but possibly with approximate variance when seen as Beta/Dirichlet probability distributions. All other operators produce opinions where the expectation value and the variance are always equal to the analytically correct values.

Different composite propositions that traditionally are equivalent in propositional logic do not necessarily have equal opinions. For example, in general

$$\omega_{x\wedge(y\vee z)} \neq \omega_{(x\wedge y)\vee(x\wedge z)} \tag{4.1}$$

although the distributivity of conjunction over disjunction, which in binary propositional logic is expressed as $x \wedge (y \vee z) \Leftrightarrow (x \wedge y) \vee (x \wedge z)$ holds. This is no surprise as the corresponding probability operator multiplication is non-distributive on comultiplication. However, multiplication is distributive over addition, as expressed by

$$\omega_{x\wedge(y\cup z)} = \omega_{(x\wedge y)\cup(x\wedge z)} \; . \tag{4.2}$$

De Morgan's laws are also satisfied as e.g. expressed by

$$\omega_{\overline{x\wedge y}} = \omega_{\overline{x}\vee\overline{y}} \; . \tag{4.3}$$

Subjective logic provides of a set of operators where input and output arguments are in the form of opinions. Opinions can be applied to frames of any cardinality, but some subjective logic operators are only defined for binomial opinions defined over binary frames. Opinion operators can be described for the belief notation, for the evidence notation, or for in the probabilistic notation, but operators defined for the belief notation of opinions are normally the simplest and most compact.

Table 4.1 provides the equivalent values and interpretation in belief notation, evidence notation, and probabilistic notation as well as in binary logic and traditional probability representation for a small set of binomial opinions.

It can be seen that some values correspond to binary logic and probability values, whereas other values correspond to probability density distributions. This richness of expression represents the advantage of subjective logic over other probabilistic logic frameworks. Online visualisations of subjective opinions and density functions can be accessed at `http://folk.uio.no/josang/sl/`.

Subjective logic allows extremely efficient computation of mathematically complex models. This is possible by approximating the analytical function expressions whenever needed. While it is relatively simple to analytically multiply two Beta distributions in the form of a joint distribution, anything more complex than that quickly becomes intractable. When combining two Beta distributions with some operator/connective, the analytical result

Table 4.1: Example values with the three equivalent notations of binomial opinion, and their interpretations.

Belief (b, d, u, a)	Evidence (r, s, a)	Probabilistic (E, c, a)	Equivalent interpretation in binary logic and/or as probability value.
$(1, 0, 0, a)$	$(\infty, 0, a)$	$(1, 1, a)$	Binary logic TRUE, and probability $p = 1$
$(0, 1, 0, a)$	$(0, \infty, a)$	$(0, 1, a)$	Binary logic FALSE, and probability $p = 0$
$(0, 0, 1, a)$	$(0, 0, a)$	$(a, 0, a)$	Vacuous opinion, Beta density with prior a
$(\frac{1}{2}, \frac{1}{2}, 0, a)$	(∞, ∞, a)	$(\frac{1}{2}, 1, a)$	Dogmatic opinion, probability $p = \frac{1}{2}$, Dirac delta function with (irrelevant) prior a
$(0, 0, 1, \frac{1}{2})$	$(0, 0, \frac{1}{2})$	$(\frac{1}{2}, 0, \frac{1}{2})$	Vacuous opinion, uniform Beta distribution over binary frame
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$	$(1, 1, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	Symmetric Beta density after 1 positive and 1 negative observation, binary frame

is not always a Beta distribution and can involve hypergeometric series. In such cases, subjective logic always approximates the result as an opinion that is equivalent to a Beta distribution.

4.2 Overview of Subjective Logic Operators

Table 4.2 provides a brief overview of the main subjective logic operators. Additional operators exist for modeling special situations, such as when fusing opinions of multiple observers. Most of the operators correspond to well-known operators from binary logic and probability calculus, whereas others are specific to subjective logic.

Table 4.2: Correspondence between probability, set and logic operators.

Subjective logic operator	Symbol	Binary logic/ set operator	Symbol	Subjective logic notation
Addition[20]	$+$	Union	$\cup$	$\omega_{x\cup y} = \omega_x + \omega_y$
Subtraction[20]	$-$	Difference	$\setminus$	$\omega_{x\setminus y} = \omega_x - \omega_y$
Multiplication[13]	$\cdot$	AND	$\wedge$	$\omega_{x\wedge y} = \omega_x \cdot \omega_y$
Division[13]	$/$	UN-AND	$\overline{\wedge}$	$\omega_{x\overline{\wedge} y} = \omega_x / \omega_y$
Comultiplication[13]	$\sqcup$	OR	$\vee$	$\omega_{x\vee y} = \omega_x \sqcup \omega_y$
Codivision[13]	$\overline{\sqcup}$	UN-OR	$\overline{\vee}$	$\omega_{x\overline{\vee} y} = \omega_x \overline{\sqcup} \omega_y$
Complement[6]	$\neg$	NOT	$\overline{x}$	$\omega_{\overline{x}} = \neg\omega_x$
Deduction[9, 18]	$\circledcirc$	MP	$\parallel$	$\omega_{Y\parallel X} = \omega_X \circledcirc \omega_{Y\mid X}$
Abduction[9, 22]	$\overline{\circledcirc}$	MT	$\overline{\parallel}$	$\omega_{Y\overline{\parallel} X} = \omega_X \overline{\circledcirc} \omega_{X\mid Y}$
Discounting[15]	$\otimes$	Transitivity	$:$	$\omega_x^{A:B} = \omega_B^A \otimes \omega_x^B$
Cumulative Fusion[15]	$\oplus$	n.a.	$\diamond$	$\omega_X^{A\diamond B} = \omega_X^A \oplus \omega_X^B$
Cumulative Unfusion[10]	$\ominus$	n.a.	$\overline{\diamond}$	$\omega_X^{A\overline{\diamond} B} = \omega_X^A \ominus \omega_X^B$
Averaging Fusion[15]	$\underline{\oplus}$	n.a.	$\underline{\diamond}$	$\omega_x^{A\underline{\diamond} B} = \omega_x^A \underline{\oplus} \omega_x^B$
Averaging Unfusion[10]	$\underline{\ominus}$	n.a.	$\underline{\overline{\diamond}}$	$\omega_X^{A\underline{\overline{\diamond}} B} = \omega_X^A \underline{\ominus} \omega_X^B$
Belief Constraining[16, 17]	$\odot$	n.a.	&	$\omega_X^{A\&B} = \omega_X^A \odot \omega_X^B$

Subjective logic is a generalisation of binary logic and probability calculus. This means that when a corresponding operator exists in binary logic, and the input parameters are equivalent to binary logic TRUE or FALSE, then the result opinion is equivalent to the result that the corresponding binary logic expression would have produced.

We will consider the case of binary logic AND which corresponds to multiplication of opinions [13]. For example, the pair of binomial opinions (in probabilistic notation) $\omega_x = (1, 1, a_x)$ and $\omega_y = (0, 1, a_y)$ produces $\omega_{x \wedge y} = (0, 1, a_x a_y)$ which is equivalent to TRUE $\wedge$ FALSE = FALSE.

Similarly, when a corresponding operator exists in probability calculus, then the probability expectation value of the result opinion is equal to the result that the corresponding probability calculus expression would have produced with input arguments equal to the probability expectation values of the input opinions.

For example, the pair of argument opinions (in probabilistic notation): $\omega_x = (\mathrm{E}_x, 1, a_x)$ and $\omega_y = (\mathrm{E}_y, 1, a_y)$ produces $\omega_{x \wedge y} = (\mathrm{E}_x \mathrm{E}_y, 1, a_x a_y)$ which is equivalent to $p(x \wedge y) = p(x)p(y)$.

It is interesting to note that subjective logic represents a calculus for Dirichlet distributions when opinions are equivalent to Dirichlet distributions. Analytical manipulations of Dirichlet distributions is complex but can be done for simple operators, such as multiplication in which case it is called a joint distribution. However, this analytical method will quickly become unmanageable when applied to the more complex operators of Table 4.2 such as conditional deduction and abduction. Subjective logic therefore has the advantage of providing advanced operators for Dirichlet distributions for which no practical analytical solutions exist. It should be noted that the simplicity of some subjective logic operators comes at the cost of allowing those operators to be approximations of the analytically correct operators. This is discussed in more detail in Sec.4.4.1.

The next sections briefly describe the operators mentioned in Table 4.2. Online demonstrations of subjective logic operators can be accessed at
`http://folk.uio.no/josang/sl/`.

4.3 Addition and Subtraction

The addition of opinions in subjective logic is a binary operator that takes opinions about two mutually exclusive alternatives (*i.e.* two disjoint subsets of the same frame) as arguments, and outputs an opinion about the union of the subsets. The operator for addition first described in [20] is defined below.

Definition 15 (Addition) *Let x and y be two disjoint subsets of the same frame X, i.e. $x \cap y = \emptyset$. The opinion about $x \cup y$ as a function of the opinions about x and y is defined as:*

$$\text{Sum } \omega_{x\cup y}: \begin{cases} b_{x\cup y} &= b_x + b_y, \\ d_{x\cup y} &= \frac{a_x(d_x - b_y) + a_y(d_y - b_x)}{a_x + a_y}, \\ u_{x\cup y} &= \frac{a_x u_x + a_y u_y}{a_x + a_y}, \\ a_{x\cup y} &= a_x + a_y. \end{cases} \tag{4.4}$$

By using the symbol "+" to denote the addition operator for opinions, addition can be denoted as $\omega_{x\cup y} = \omega_x + \omega_y$.

The inverse operation to addition is subtraction. Since addition of opinions yields the opinion about $x \cup y$ from the opinions about disjoint subsets of the frame, then the difference between the opinions about x and y (*i.e.* the opinion about $x \backslash y$) can only be defined if $y \subseteq x$ where x and y are being treated as subsets of the frame X, *i.e.* the system must be in the state x whenever it is in the state y. The operator for subtraction first described in [20] is defined below.

Definition 16 (Subtraction) *Let x and y be subsets of the same frame X so that x and y, i.e. $x \cap y = y$. The opinion about $x \backslash y$ as a function of the opinions about x and y is defined as:*

The opinion about $x \backslash y$ is given by

$$\text{Difference } \omega_{x\backslash y}: \begin{cases} b_{x\backslash y} &= b_x - b_y, \\ d_{x\backslash y} &= \frac{a_x(d_x + b_y) - a_y(1 + b_y - b_x - u_y)}{a_x - a_y}, \\ u_{x\backslash y} &= \frac{a_x u_x - a_y u_y}{a_x - a_y}, \\ a_{x\backslash y} &= a_x - a_y. \end{cases} \tag{4.5}$$

Since $u_{x\backslash y}$ should be nonnegative, then this requires that $a_y u_y \leq a_x u_x$, and since $d_{x\backslash y}$ should be nonnegative, then this requires that $a_x(d_x + b_y) \geq a_y(1 + b_y - b_x - u_y)$.

By using the symbol "−" to denote the subtraction operator for opinions, subtraction can be denoted as $\omega_{x\backslash y} = \omega_x - \omega_y$.

4.4 Binomial Multiplication and Division

This section describes the subjective logic operators that correspond to binary logic AND and OR, as well as their inverse operators. We will here describe normal multiplication and comultiplication [13] which are different from simple multiplication and comultiplication [6]. Special limit cases are described in [13].

4.4.1 Binomial Multiplication and Comultiplication

Binomial multiplication and comultiplication in subjective logic take binomial opinions about two elements from distinct binary frames of discernment as input arguments and produce a binomial opinion as result. The product and coproduct result opinions relate to subsets of the Cartesian product of the two binary frames of discernment. The Cartesian product of the two binary frames of discernment $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ produces the quaternary set $X \times Y = \{(x, y),\ (x, \overline{y}),\ (\overline{x}, y),\ (\overline{x}, \overline{y})\}$ which is illustrated in Fig.4.1 below.

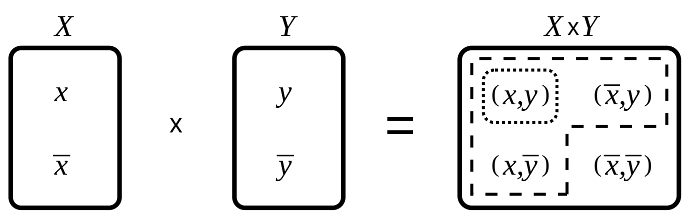


Figure 4.1: Cartesian product of two binary frames of discernment

As will be explained below, binomial multiplication and comultiplication in subjective logic represent approximations of the analytically correct product and coproducts of Beta probability density functions. In this regard, normal multiplication and comultiplication produce the best approximations.

Let ω_x and ω_y be opinions about x and y respectively held by the same observer. Then the product opinion $\omega_{x \wedge y}$ is the observer's opinion about the conjunction $x \wedge y = \{(x, y)\}$ that is represented by the area inside the dotted line in Fig.4.1. The coproduct opinion $\omega_{x \vee y}$ is the opinion about the disjunction $x \vee y = \{(x, y),\ (x, \overline{y}),\ (\overline{x}, y)\}$ that is represented by the area inside the dashed line in Fig.4.1. Obviously $X \times Y$ is not binary, and coarsening is required in order to determine the product and coproduct opinions as binomial opinions.

Definition 17 (Normal Binomial Multiplication) *Let $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ be two separate frames, and let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_x = (b_y, d_y, u_y, a_y)$ be independent binomial opinions on x and y respectively. Given opinions about independent propositions, x and y, the binomial opinion $\omega_{x \wedge y}$ on the conjunction $(x \wedge y)$ is given by*

$$\text{Product } \omega_{x \wedge y} : \begin{cases} b_{x \wedge y} &= b_x b_y + \frac{(1-a_x)a_y b_x u_y + a_x(1-a_y)u_x b_y}{1-a_x a_y}, \\ \\ d_{x \wedge y} &= d_x + d_y - d_x d_y, \\ \\ u_{x \wedge y} &= u_x u_y + \frac{(1-a_y)b_x u_y + (1-a_x)u_x b_y}{1-a_x a_y}, \\ \\ a_{x \wedge y} &= a_x a_y. \end{cases} \tag{4.6}$$

By using the symbol "·" to denote this operator multiplication of opinions can be written as $\omega_{x \wedge y} = \omega_x \cdot \omega_y$.

Definition 18 (Normal Binomial Comultiplication) *Let $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ be two separate frames, and let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_x = (b_y, d_y, u_y, a_y)$ be independent binomial opinions on x and y respectively. The binomial opinion $\omega_{x\vee y}$ on the disjunction $x \vee y$ is given by*

$$\text{Coproduct } \omega_{x\vee y} : \begin{cases} b_{x\vee y} &= b_x + b_y - b_x b_y, \\ d_{x\vee y} &= d_x d_y + \frac{a_x(1-a_y)d_x u_y + (1-a_x)a_y u_x d_y}{a_x + a_y - a_x a_y}, \\ u_{x\vee y} &= u_x u_y + \frac{a_y d_x u_y + a_x u_x d_y}{a_x + a_y - a_x a_y}, \\ a_{x\vee y} &= a_x + a_y - a_x a_y. \end{cases} \tag{4.7}$$

By using the symbol "$\sqcup$" to denote this operator multiplication of opinions can be written as $\omega_{x\vee y} = \omega_x \sqcup \omega_y$.

Normal multiplication and comultiplication represent a self-dual system represented by $b \leftrightarrow d$, $u \leftrightarrow u$, $a \leftrightarrow 1-a$, and $\wedge \leftrightarrow \vee$, that is, for example, the expressions for $b_{x\wedge y}$ and $d_{x\vee y}$ are dual to each other, and one determines the other by the correspondence, and similarly for the other expressions. This is equivalent to the observation that the opinions satisfy de Morgan's Laws, *i.e.* $\omega_{x\wedge y} = \omega_{\overline{\overline{x}\vee\overline{y}}}$ and $\omega_{x\vee y} = \omega_{\overline{\overline{x}\wedge\overline{y}}}$. However it should be noted that multiplication and comultiplication are not distributive over each other, *i.e.* for example that:

$$\omega_{x\wedge(y\vee z)} \neq \omega_{(x\wedge y)\vee(x\wedge z)} \tag{4.8}$$

This is to be expected because if x, y and z are independent, then $x \wedge y$ and $x \wedge z$ are not generally independent in probability calculus so that distributivity does not hold. In fact distributivity of conjunction over disjunction and vice versa only holds in binary logic.

Normal multiplication and comultiplication produce very good approximations of the analytically correct products and coproducts when the arguments are Beta probability density functions [13]. The difference between the subjective logic product and the analytically correct product of Beta density functions is best illustrated with the example of multiplying two equal vacuous binomial opinions $\omega = (0, 0, 1, \frac{1}{2})$, that are equivalent to the uniform Beta density functions Beta$(1, 1)$.

Theorem 2 *Let Q and R be independent random probability variables with identical uniform distributions over $[0, 1]$, which for example can be described as the Beta distribution Beta$(1, 1)$. Then the probability distribution function for the product random variable $P = QR$ is given by $f(p) = -\ln p$ for $0 < p < 1$.*

The proof is given in [13]. This result applies to the case of the independent propositions x and y, where we are taking four exhaustive and mutually exclusive propositions ($x \wedge y$, $x \wedge \overline{y}$, $\overline{x} \wedge y$, $\overline{x} \wedge \overline{y}$). Specifically, this means that when the probabilities of x and y have uniform distributions, then the probability of the conjunction $x \wedge y$ has the probability distribution function $f(p) = -\ln p$ with probability expectation value $\frac{1}{4}$.

This can be contrasted with the *a priori* non-informative probability distribution over four exhaustive and mutually exclusive propositions x_1, x_2, x_3, x_4, which can be descried by: Dirichlet $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, so that the *a priori* probability distribution for the probability of x_1 is Beta $\left(\frac{1}{2}, \frac{3}{2}\right)$, again with probability expectation value $\frac{1}{4}$.

The difference between Beta $\left(\frac{1}{2}, \frac{3}{2}\right)$, which is derivable from Dirichlet $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, and $-\ln p$ is illustrated in Fig.4.2 below.

The analytically correct product of two uniform distributions is represented by $-\ln p$, whereas the product produced by the multiplication operator is Beta $\left(\frac{1}{2}, \frac{3}{2}\right)$, which illustrates that multiplication and comultiplication in subjective logic produce approximate results. More specifically, it can be shown that the probability expectation value is always exact, and that the variance is approximate. The quality of the variance approximation is analysed in [13], and is very good in general. The discripencies grow with the amount of uncertainty in the arguments, so Fig.4.2 illustrates the worst case.

The advantage of the multiplication and comultiplication operators of subjective logic is their simplicity, which means that complex models can be analysed efficiently. The analytical result of products and coproducts of Beta distributions will in general involve the Gauss hypergeometric function [23]. The analysis of anything but the most basic models based on such functions would quickly become unmanageable.

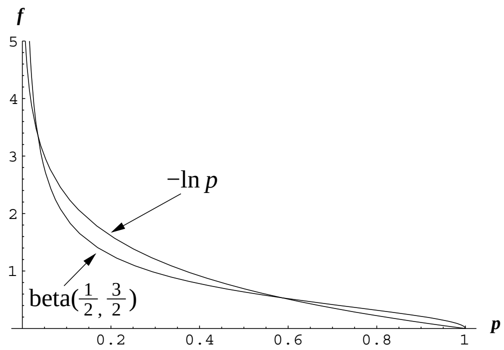


Figure 4.2: Comparison between Beta $\left(\frac{1}{2},\frac{3}{2}\right)$ and product of uniform distributions.

4.4.2 Binomial Division and Codivision

The inverse operation to binomial multiplication is binomial division. The quotient of opinions about propositions x and y represents the opinion about a proposition z which is independent of y such that $\omega_x = \omega_{y\wedge z}$. This requires that:

$$\begin{cases} a_x < a_y\ , \\ d_x \geq d_y\ , \\ b_x \geq \frac{a_x(1-a_y)(1-d_x)b_y}{(1-a_x)a_y(1-d_y)}\ , \\ u_x \geq \frac{(1-a_y)(1-d_x)u_y}{(1-a_x)(1-d_y)}\ . \end{cases} \tag{4.9}$$

Definition 19 (Normal Binomial Division) *Let $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ be frames, and let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_y = (b_y, d_y, u_y, a_y)$ be binomial opinions on x and y satisfying Eq.(4.9). The division of ω_x by ω_y produces the quotient opinion $\omega_{x\overline{\wedge}y} = (b_{x\overline{\wedge}y}, d_{x\overline{\wedge}y}, u_{x\overline{\wedge}y}, a_{x\overline{\wedge}y})$ defined by*

$$\text{Quotient } \omega_{x\overline{\wedge}y} : \begin{cases} b_{x\overline{\wedge}y} &= \frac{a_y(b_x+a_xu_x)}{(a_y-a_x)(b_y+a_yu_y)} - \frac{a_x(1-d_x)}{(a_y-a_x)(1-d_y)}, \\ \\ d_{x\overline{\wedge}y} &= \frac{d_x-d_y}{1-d_y}, \\ \\ u_{x\overline{\wedge}y} &= \frac{a_y(1-d_x)}{(a_y-a_x)(1-d_y)} - \frac{a_y(b_x+a_xu_x)}{(a_y-a_x)(b_y+a_yu_y)}, \\ \\ a_{x\overline{\wedge}y} &= \frac{a_x}{a_y}, \end{cases} \tag{4.10}$$

By using the symbol "/" to denote this operator, division of opinions can be written as $\omega_{x\overline{\wedge}y} = \omega_x/\omega_y$.

The inverse operation to comultiplication is codivision. The co-quotient of opinions about propositions x and y represents the opinion about a proposition z which is independent of y such that $\omega_x = \omega_{y\vee z}$. This requires that

$$\begin{cases} a_x > a_y\ , \\ b_x \geq b_y\ , \\ d_x \geq \frac{(1-a_x)a_y(1-b_x)d_y}{a_x(1-a_y)(1-b_y)}\ , \\ u_x \geq \frac{a_y(1-b_x)u_y}{a_x(1-b_y)}\ . \end{cases} \tag{4.11}$$

Definition 20 (Normal Binomial Codivision) *Let $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ be frames, and let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_y = (b_y, d_y, u_y, a_y)$ be binomial opinions on x and y satisfying Eq.(4.11). The codivision of opinion ω_x by* *opinion ω_x produces the co-quotient opinion $\omega_{x\overline{\vee}y} = (b_{x\overline{\vee}y}, d_{x\overline{\vee}y}, u_{x\overline{\vee}y}, a_{x\overline{\vee}y})$ defined by*

$$\text{Co-quotient } \omega_{x\overline{\vee}y}: \begin{cases} b_{x\overline{\vee}y} &= \frac{b_x - b_y}{1-b_y}, \\ d_{x\overline{\vee}y} &= \frac{(1-a_y)(d_x+(1-a_x)u_x)}{(a_x-a_y)(d_y+(1-a_y)u_y)} - \frac{(1-a_x)(1-b_x)}{(a_x-a_y)(1-b_y)}, \\ u_{x\overline{\vee}y} &= \frac{(1-a_y)(1-b_x)}{(a_x-a_y)(1-b_y)} - \frac{(1-a_y)(d_x+(1-a_x)u_x)}{(a_x-a_y)(d_y+(1-a_y)u_y)}, \\ a_{x\overline{\vee}y} &= \frac{a_x-a_y}{1-a_y}, \end{cases} \tag{4.12}$$

By using the symbol "$\overline{\sqcup}$" to denote this operator, codivision of opinions can be written as $\omega_{x\overline{\vee}y} = \omega_x \overline{\sqcup} \omega_y$.

4.4.3 Correspondence to Other Logic Frameworks

Multiplication, comultiplication, division and codivision of dogmatic opinions are equivalent to the corresponding probability calculus operators in Table 4.3, where e.g. $p(x)$ denotes the probability of the state variable x.

Operator name:	Probability calculus operators	
Multiplication	$p(x \wedge y)$	$= p(x)p(y)$
Division	$p(x \overline{\wedge} y)$	$= p(x)/p(y)$
Comultiplication	$p(x \vee y)$	$= p(x) + p(y) - p(x)p(y)$
Codivision	$p(x \overline{\vee} y)$	$= (p(x) - p(y))/(1 - p(y))$

Table 4.3: Probability calculus operators corresponding to opinion operators.

In the case of absolute opinions, i.e. when either $b = 1$ (absolute belief) or $d = 1$ (absolute disbelief), then the multiplication and comultiplication operators are equivalent to the AND and OR operators of binary logic.

4.5 Multinomial Multiplication

Multinomial multiplication is different from binomial multiplication in that the product opinion on the whole product frame is considered, instead of just on one element of the product frame. Fig.4.3 below illustrates the general situation.

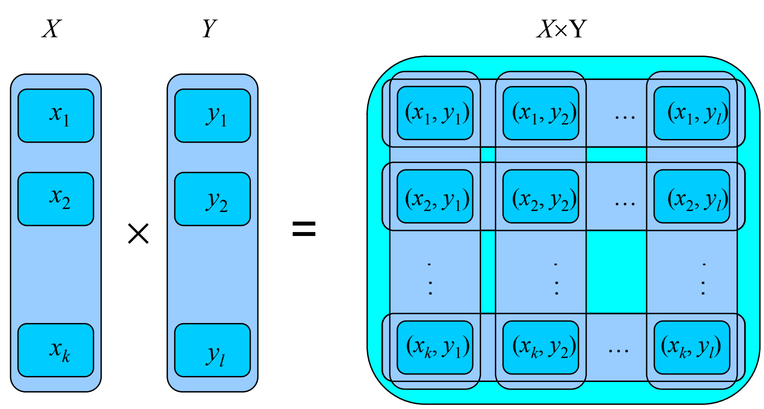


Figure 4.3: Cartesian product of two n-ary frames

Assuming the opinions ω_X and ω_Y, and the product of every combination of belief mass and uncertainty mass from the two opinions as intermediate products, there will be belief mass on all the shaded subsets of the product frame. In order to produce an opinion with only belief mass on each singleton element of $X \times Y$ as well as on $X \times Y$ itself, some of the belief mass on the row and column subsets of $X \times Y$ must be redistributed to the singleton elements in such a way that the expectation value of each singleton element equals the product of the expectation values of pairs of singletons from X and Y respectively.

4.5.1 General Approach

Evaluating the products of two separate multinomial opinions involves the Cartesian product of the respective frames to which the opinions apply. Let ω_X and ω_Y be two independent multinomial opinions that apply to the separate frames

$$X = \{x_1, x_2, \dots x_k\} \text{ with cardinality } k \tag{4.13}$$

$$Y = \{y_1, y_2, \dots y_l\} \text{ with cardinality } l\,. \tag{4.14}$$

The Cartesian product $X \times Y$ with cardinality kl is expressed as the matrix:

$$X \times Y = \begin{pmatrix} (x_1, y_1), & (x_2, y_1), & \cdots & (x_k, y_1) \\ (x_1, y_2), & (x_2, y_2), & \cdots & (x_k, y_2) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (x_1, y_l), & (x_2, y_l), & \cdots & (x_k, y_l) \end{pmatrix} \tag{4.15}$$

We now turn to the product of the multinomial opinions. The raw terms produced by $\omega_X \cdot \omega_Y$ can be separated into different groups.

1. The first group of terms consists of belief masses on singletons of $X \times Y$:

$$b^{\mathrm{I}}_{X \times Y} = \begin{cases} b_X(x_1)b_Y(y_1), & b_X(x_2)b_Y(y_1), & \dots & b_X(x_k)b_Y(y_1) \\ b_X(x_1)b_Y(y_2), & b_X(x_2)b_Y(y_2), & \dots & b_X(x_k)b_Y(y_2) \\ . & . & \dots & . \\ b_X(x_1)b_Y(y_l), & b_X(x_2)b_Y(y_l), & \dots & b_X(x_k)b_Y(y_l) \end{cases} \tag{4.16}$$

2. The second group of terms consists of belief masses on rows of $X \times Y$:

$$b^{\mathrm{Rows}}_{X \times Y} = \begin{pmatrix} u_X b_Y(y_1), & u_X b_Y(y_2), & \dots & u_X b_Y(y_l) \end{pmatrix} \tag{4.17}$$

3. The third group of terms consists of belief masses on columns of $X \times Y$:

$$b^{\mathrm{Columns}}_{X \times Y} = \begin{pmatrix} b_X(x_1)u_Y, & b_X(x_2)u_Y, & \dots & b_X(x_k)u_Y \end{pmatrix} \tag{4.18}$$

4. The last term is simply the belief mass on the whole product frame:

$$u^{\mathrm{Frame}}_{X \times Y} = u_X u_Y \tag{4.19}$$

The singleton terms of Eq.(4.16) and the term on the whole frame are unproblematic because they conform with the opinion representation of having belief mass only on singletons and on the whole frame. In contrast, the terms on rows and columns apply to overlapping subsets which is not compatible with the required opinion format, and therefore need to be reassigned. Some of it can be reassigned to singletons, and some to the whole frame. There are several possible strategies for determining the amount of uncertainty mass to be assigned to singletons and to the frame. Two methods are described below.

4.5.2 Determining Uncertainty Mass

1. **The Method of Assumed Belief Mass:** The simplest method is to assign the belief mass from the terms of Eq.(4.17) and Eq.(4.18) to singletons. Only the uncertainty mass from Eq.(4.19) is then considered as uncertainty in the product opinion, expressed as:

$$u_{X \times Y} = u_X u_Y \;. \tag{4.20}$$

A problem with this approach is that it in general produces less uncertainty than intuition would dictate.

2. **The Method of Assumed Uncertainty Mass:** A method that preserves more uncertainty is to consider the belief mass from Eq.(4.17) and Eq.(4.18) as potential uncertainty mass that together with the uncertainty mass from Eq.(4.19) can be called intermediate uncertainty mass. The intermediate uncertainty mass is thus:

$$u^{\mathrm{I}}_{X \times Y} = u^{\mathrm{Rows}}_{X \times Y} + u^{\mathrm{Columns}}_{X \times Y} + u^{\mathrm{Frame}}_{X \times Y} \tag{4.21}$$

The probability expectation values of each singleton in the product frame can easily be computed as the product of the expectation values of each pair of states from X and Y, as expressed in Eq.(4.22).

$$\begin{aligned} \mathrm{E}((x_i, y_j)) &= \mathrm{E}(x_i)\mathrm{E}(y_j) \\ &= (b_X(x_i) + a_X(x_i)u_X)(b_Y(y_j) + a_Y(y_j)u_Y) \end{aligned} \tag{4.22}$$

We also require that the probability expectation values of the states in the product frame can be computed as a function of the product opinion according to Eq.(4.23).

$$\mathrm{E}((x_i, y_j)) = b_{X \times Y}((x_i, y_j)) + a_X(x_i)a_Y(y_j)u_{X \times Y} \tag{4.23}$$

In order to find the correct uncertainty mass for the product opinion, each state $(x_i, y_j) \in X \times Y$ will be investigated in turn to find the smallest uncertainty mass that satisfies both Eq.(4.23) and Eq.(4.24).

$$\frac{b^{\mathrm{I}}_{X \times Y}((x_i, y_j))}{u^{\mathrm{I}}_{X \times Y}} = \frac{b_{X \times Y}((x_i, y_j))}{u_{X \times Y}} \tag{4.24}$$

The uncertainty mass that satisfies both Eq.(4.23) and Eq.(4.24) for state (x_i, y_j) can be expressed as:

$$u^{(i,j)}_{X \times Y} = \frac{u^{\mathrm{I}}_{X \times Y}\mathrm{E}((x_i, y_j))}{b^{\mathrm{I}}_{X \times Y}((x_i, y_j)) + a_X(x_i)a_Y(y_j)u^{\mathrm{I}}_{X \times Y}} \tag{4.25}$$

The product uncertainty can now be determined as the smallest $u^{(i,j)}_{X \times Y}$ among all the states, expressed as:

$$u_{X \times Y} = \min\left\{u^{(i,j)}_{X \times Y} \;\; \text{where} \;\; (x_i, y_j) \in X \times Y\right\} \tag{4.26}$$

4.5.3 Determining Belief Mass

Having determined the uncertainty mass, either according to Eq.(4.20) or according to Eq.(4.26), the expression for the product expectation of Eq.(4.22) can be used to compute the belief mass on each element in the product frame, as expressed by Eq.(4.27).

$$b_{X \times Y}((x_i, y_j)) = \mathrm{E}((x_i, y_j)) - a_X(x_i)a_Y(y_j)u_{X \times Y} \tag{4.27}$$

It can be shown that the additivity property of Eq.(4.28) is preserved.

$$u_{X \times Y} + \sum_{(x_i, y_j) \in X \times Y} b_{X \times Y}((x_i, y_j)) = 1 \tag{4.28}$$

From Eq.(4.27) it follows directly that the product operator is commutative. It can also be shown that the product operator is associative.

4.5.4 Example

We consider the scenario where a GE (Genetic Engineering) process can produce Male (M) or Female (F) eggs, and that in addition, each egg can have genetical mutation S or T independently of its gender. This constitutes two binary frames $X = \{M, F\}$ and $Y = \{S, T\}$, or alternatively the quaternary product frame $X \times Y = \{MS, MT, FS, FT\}$. Sensor A observes whether each egg is M or F, and Sensor B observes whether the egg has mutation S or T.

Assume that an opinion regarding the gender of a specific egg is derived from Sensor A data, and that an opinion regarding its mutation is derived from Sensor B data. Sensors A and Sensor B have thus observed different and orthogonal aspects, so their respective opinions can be combined with multiplication. This is illustrated in Fig.4.4.

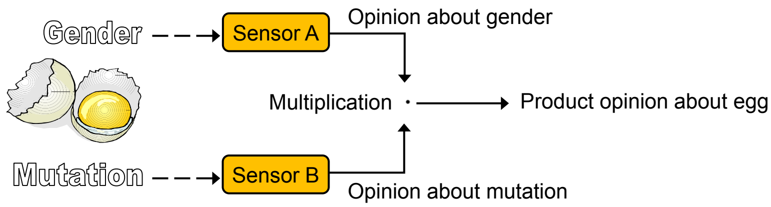


Figure 4.4: Multiplication of opinions on orthogonal aspects of GE eggs

The result of the opinion multiplication can be considered as an opinion based on a single observation where both aspects are observed at the same time. Let the observation opinions be:

$$\text{Gender } \omega_X^A : \begin{cases} \vec{b}_X^A = (0.8, 0.1) \\ u_X^A = 0.1 \\ \vec{a}_X^A = (0.5, 0.5) \end{cases} \quad \text{Mutation } \omega_Y^B : \begin{cases} \vec{b}_Y^B = (0.7, 0.1) \\ u_Y^B = 0.2 \\ \vec{a}_Y^B = (0.2, 0.8) \end{cases} \tag{4.29}$$

The Cartesian product frame can be expressed as:

$$X \times Y = \begin{pmatrix} MS, & FS \\ MT, & FT \end{pmatrix} \tag{4.30}$$

According to Eq.(4.22) the product expectation values are:

$$E(X \times Y) = \begin{pmatrix} 0.629, & 0.111 \\ 0.221, & 0.039 \end{pmatrix} \tag{4.31}$$

Below are described the results of both methods proposed in Sec.4.5.2.

1. When applying the method of *Assumed Belief Mass* where the uncertainty mass is determined according to Eq.(4.20), the product opinion is computed as:

$$b_{X \times Y} = \begin{pmatrix} 0.627, & 0.109 \\ 0.213, & 0.031 \end{pmatrix}, \quad u_{X \times Y} = 0.02, \quad a_{X \times Y} = \begin{pmatrix} 0.1, & 0.4 \\ 0.1, & 0.4 \end{pmatrix} \tag{4.32}$$

2. When applying the method of *Assumed Uncertainty* where the uncertainty mass is determined according to Eq.(4.25) and Eq.(4.26), the product opinion is computed as:

$$b_{X \times Y} = \begin{pmatrix} 0.620, & 0.102 \\ 0.185, & 0.003 \end{pmatrix}, \quad u_{X \times Y} = 0.09, \quad a_{X \times Y} = \begin{pmatrix} 0.1, & 0.4 \\ 0.1, & 0.4 \end{pmatrix} \tag{4.33}$$

The results indicate that there can be a significant difference between the two methods, and that the safest approach is to use the *assumed uncertainty* method because it preserves the most uncertainty in the product opinion.

4.6 Deduction and Abduction

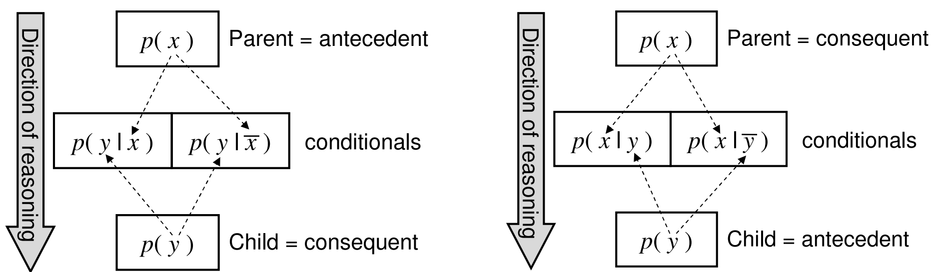
Both binary logic and probability calculus have mechanisms for conditional reasoning. In binary logic, Modus Ponens (MP) and Modus Tollens (MT) are the classical operators which are used in any field of logic that requires conditional deduction. In probability calculus, conditional probabilities together with base rates are used for analysing deductive and abductive reasoning models. Subjective logic extends the traditional probabilistic approach by allowing subjective opinions to be used as input arguments, so that deduced or abduced conclusions reflect the underlying uncertainty of the situation.

4.6.1 Probabilistic Deduction and Abduction

In order to clarify the principles of deduction and abduction, this section provides a brief overview of deduction and abduction in traditional probability calculus.

The notation $y\|x$, introduced in [18], denotes that the truth or probability of proposition y is derived as a function of the probability of the antecedent x together with the conditionals $p(y|x)$ and $p(y|\overline{x})$. The expression $p(y\|x)$ thus represents a derived value, whereas the expression $p(y|x)$ represents an input argument. This notational convention will also be used in subjective logic.

The deductive and abductive reasoning situations are illustrated in Fig.4.5 where x denotes the parent state and y denotes the child state of the reasoning. Conditionals are expressed as p(consequent |antecedent), i.e. with the consequent first, and the antecedent last.



(a) Deduction. (b) Abduction.

Figure 4.5: Visualising deduction and abduction

It is assumed that the analyst has evidence about the parent, and wants to derive an opinion about the child. Defining parent and child is thus equivalent with defining the reasoning direction.

Forward conditional inference, called *deduction*, is when the parent and child states of the reasoning are the antecedent and consequent states respectively of the available conditionals.

Assume that x and $\overline{x}$ are relevant to y according to the conditional statements $y|x$ and $y|\overline{x}$, where x and $\overline{x}$ are antecedents and y is the consequent of the conditionals. Let $p(x)$, $p(y|x)$ and $p(y|\overline{x})$ be probability assessments of x, $y|x$ and $y|\overline{x}$ respectively. The conditionally deduced probability $p(y\|x)$ can then be computed as:

$$p(y\|x) = p(x)p(y|x) + p(\overline{x})p(y|\overline{x}) \qquad = p(x)p(y|x) + (1 - p(x))p(y|\overline{x}) \ . \tag{4.34}$$

Reverse conditional inference, called *abduction*, is when the parent state of the reasoning is the consequent of the conditionals, and the child state of the reasoning is the antecedent state of the conditionals.

Assume that the states y and $\overline{y}$ are relevant to x according the conditional statements $x|y$ and $x|\overline{y}$, where y and $\overline{y}$ are antecedents and x is the consequent of the conditionals. Let $p(x)$, $p(x|y)$ and $p(x|\overline{y})$ be probability assessments of x, $x|y$ and $x|\overline{y}$ respectively. The conditionally abduced probability $p(y\overline{\|}x)$ can then be computed as:

$$p(y\overline{\|}x) = p(x)\left(\frac{a(y)p(x|y)}{a(y)p(x|y) + a(\overline{y})p(x|\overline{y})}\right) + p(\overline{x})\left(\frac{a(y)p(\overline{x}|y)}{a(y)p(\overline{x}|y) + a(\overline{y})p(\overline{x}|\overline{y})}\right) \tag{4.35}$$

It can be noted that Eq.(4.35) is simply the application of conditional deduction according to Eq.(4.34) where the conditionals have been computed as:

$$\begin{aligned} p(y|x) &= a(y)p(x|y)/(a(y)p(x|y) + a(\overline{y})p(x|\overline{y})) \\ p(y|\overline{x}) &= a(y)p(\overline{x}|y)/(a(y)p(\overline{x}|y) + a(\overline{y})p(\overline{x}|\overline{y})) \end{aligned} \tag{4.36}$$

The terms used in Eq.(4.34), Eq.(4.35) and Eq.(4.36) are interpreted as follows:

$p(y|x)$	:	the conditional probability of y given that x is TRUE
$p(y|\overline{x})$	:	the conditional probability of y given that x is FALSE
$p(x|y)$	:	the conditional probability of x given that y is TRUE
$p(x|\overline{y})$	:	the conditional probability of x given that y is FALSE
$p(x)$	:	the probability of x
$p(\overline{x})$	:	the probability of the complement of x ($= 1 - p(x)$)
$a(y)$	:	the base rate of y
$p(y\|x)$	:	the deduced probability of y as a function of observation x
$p(y\overline{\|}x)$	:	the abduced probability of the consequent y as a function of observation x

The binomial expressions for probabilistic deduction of Eq.(4.34) and probabilistic abduction of Eq.(4.35) can be generalised to multinomial expressions as explained below.

Let $X = \{x_i|i = 1 \ldots k\}$ be the parent frame with cardinality k, and let $Y = \{y_j|j = 1 \ldots l\}$ be the child frame with cardinality l. The deductive conditional relationship between X and Y is then expressed with k vector conditionals $p(Y|x_i)$, each being of l dimensions. This is illustrated in Fig.4.6.

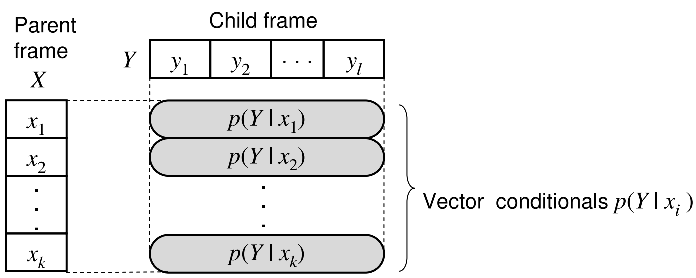


Figure 4.6: Multinomial deductive vector conditionals between parent X and child Y

The vector conditional $\vec{p}(Y|x_i)$ relates each state x_i to the frame Y. The elements of $\vec{p}(Y|x_i)$ are the scalar conditionals expressed as:

$$p(y_j|x_i), \qquad \text{where } \sum_{j=1}^{l} p(y_j|x_i) = 1\ . \tag{4.37}$$

The probabilistic expression for multinomial conditional deduction from X to Y is the vector $\vec{p}(Y\|X)$ over Y where each scalar vector element $p(y_j\|X)$ is:

$$p(y_j\|X) = \sum_{i=1}^{k} p(x_i)p(y_j|x_i)\ . \tag{4.38}$$

The multinomial probabilistic expression for inverting conditionals is:

$$p(y_j|x_i) = \frac{a(y_j)p(x_i|y_j)}{\sum_{t=1}^{l} a(y_t)p(x_i|y_t)} \tag{4.39}$$

where $a(y_j)$ represents the base rate of y_j.

By substituting the conditionals of Eq.(4.38) with inverted multinomial conditionals from Eq.(4.39), the general expression for probabilistic abduction emerges:

$$p(y_j \overline{\|} X) = \sum_{i=1}^{k} p(x_i) \left(\frac{a(y_j) p(x_i|y_j)}{\sum_{t=1}^{l} a(y_t) p(x_i|y_t)} \right) . \tag{4.40}$$

This will be illustrated by a numerical example below.

Example: Probabilistic Intelligence Analysis

Two countries *A* and *B* are in conflict, and intelligence analysts of country *B* wants to find out whether country *A* intends to use military aggression. The analysts of country *B* consider the following possible alternatives regarding country *A*'s plans:

y_1 : No military aggression from country *A*
y_2 : Minor military operations by country *A* (4.41)
y_3 : Full invasion of country *B* by country *A*

The way the analysts will determine the most likely plan of country *A* is by trying to observe movement of troops in country *A*. For this, they have spies placed inside country *A*. The analysts of country *B* consider the following possible movements of troops.

x_1 : No movement of country *A*'s troops
x_2 : Minor movements of country *A*'s troops (4.42)
x_3 : Full mobilisation of all country *A*'s troops

The analysts have defined a set of conditional probabilities of troop movements as a function of military plans, as specified by Table 4.4.

Table 4.4: Conditional probabilities $p(X|Y)$: troop movement x_i given military plan y_j

Probability vectors	Troop movements x_1 No movemt.	x_2 Minor movemt.	x_3 Full mob.
$\vec{p}(X|y_1)$:	$p(x_1|y_1) = 0.50$	$p(x_2|y_1) = 0.25$	$p(x_3|y_1) = 0.25$
$\vec{p}(X|y_2)$:	$p(x_1|y_2) = 0.00$	$p(x_2|y_2) = 0.50$	$p(x_3|y_2) = 0.50$
$\vec{p}(X|y_3)$:	$p(x_1|y_3) = 0.00$	$p(x_2|y_3) = 0.25$	$p(x_3|y_3) = 0.75$

The rationale behind the conditionals are as follows. In case country *A* has no plans of military aggression (y_1), then there is little logistic reason for troop movements. However, even without plans of military aggression against country *B* it is possible that country *A* expects military aggression from country *B*, forcing troop movements by country *A*. In case country *A* prepares for minor military operations against country *B* (y_2), then necessarily troop movements are required. In case country *A* prepares for full invasion of country *B* (y_3), then significant troop movements are required.

Based on observations by spies of country *B*, the analysts determine the likelihoods of actual troop movements to be:

$$p(x_1) = 0.00 , \quad p(x_2) = 0.50 , \quad p(x_3) = 0.50 . \tag{4.43}$$

The analysts are faced with an abductive reasoning situation and must first derive the conditionals $p(Y|X)$. The base rate of military plans is set to:

$$a(y_1) = 0.70 , \quad a(y_2) = 0.20 , \quad a(y_3) = 0.10 . \tag{4.44}$$

The expression of Eq.(4.39) can now be used to derive the required conditionals, which are given in Table 4.5 below.

Table 4.5: Conditional probabilities $p(Y|X)$: military plan y_j given troop movement x_i

	Probabilities of military plans given troop movement		
	$\vec{p}(Y|x_1)$	$\vec{p}(Y|x_2)$	$\vec{p}(Y|x_3)$
Military plan	No movemt.	Minor movemt.	Full mob.
y_1: No aggr.	$p(y_1|x_1) = 1.00$	$p(y_1|x_2) = 0.58$	$p(y_1|x_3) = 0.50$
y_2: Minor ops.	$p(y_2|x_1) = 0.00$	$p(y_2|x_2) = 0.34$	$p(y_2|x_3) = 0.29$
y_3: Invasion	$p(y_3|x_1) = 0.00$	$p(y_3|x_2) = 0.08$	$p(y_3|x_3) = 0.21$

The expression of Eq.(4.38) can now be used to derive the probabilities of military plans of country A, resulting in:

$$p(y_1\|X) = 0.54\ , \qquad p(y_2\|X) = 0.31\ , \qquad p(y_3\|X) = 0.15\ . \tag{4.45}$$

Based on the results of Eq.(4.45), it seems most likely that country A does not plan any military aggression against country B. Analysing the same example with subjective logic in Sec.4.6.3 will show that these results give a misleading estimate of country A's plans because they hide the underlying uncertainty.

4.6.2 Binomial Deduction and Abduction with Subjective Opinions

This section presents conditional deduction and abduction with binomial opinions.

Binomial Deduction

Conitional deduction with binomial opinions has previously been described in [18]. It is a generalisation of probabilistic conditional deduction expressed in Eq.(4.34).

Definition 21 (Conditional Deduction with Binomial Opinions) *Let $X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$ be two binary frames where there is a degree of relevance of X to Y. Let $\omega_x = (b_x, d_x, u_x, a_x)$, $\omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_{y|x})$ and $\omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_{y|\overline{x}})$ be an agent's respective opinions about x being true, about y being true given that x is true and about y being true given that x is false. Let $\omega_{y\|x} = (b_{y\|x}, d_{y\|x}, u_{y\|x}, a_{y\|x})$ be the opinion about y such that:*

$$\omega_{y\|x} \quad \text{is defined by:} \quad \begin{cases} b_{y\|x} = b_y^{\mathrm{I}} - a_y K \\ d_{y\|x} = d_y^{\mathrm{I}} - (1 - a_y)K \\ u_{y\|x} = u_y^{\mathrm{I}} + K \\ a_{y\|x} = a_y\ . \end{cases} \tag{4.46}$$

$$\text{where} \quad \begin{cases} b_y^{\mathrm{I}} = b_x b_{y|x} + d_x b_{y|\overline{x}} + u_x(b_{y|x} a_x + b_{y|\overline{x}}(1 - a_x)) \\ d_y^{\mathrm{I}} = b_x d_{y|x} + d_x d_{y|\overline{x}} + u_x(d_{y|x} a_x + d_{y|\overline{x}}(1 - a_x)) \\ u_y^{\mathrm{I}} = b_x u_{y|x} + d_x u_{y|\overline{x}} + u_x(u_{y|x} a_x + u_{y|\overline{x}}(1 - a_x)) \end{cases} \tag{4.47}$$

and K can be determined according to the following selection criteria:

$$\textbf{Case I:} \quad ((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}})) \vee ((b_{y|x} \le b_{y|\overline{x}}) \wedge (d_{y|x} \le d_{y|\overline{x}})) \implies K = 0 \tag{4.48}$$

$$\textbf{Case II.A.1:} \quad ((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \le d_{y|\overline{x}})) \wedge (\mathrm{E}(\omega_{y\|\widehat{x}}) \le (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x}))) \wedge (\mathrm{E}(\omega_x) \le a_x) \implies K = \frac{a_x u_x (b_y^{\mathrm{I}} - b_{y|\overline{x}})}{(b_x + a_x u_x) a_y} \tag{4.49}$$

Case II.A.2:

$$\begin{aligned}&((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) \leq (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))\\ &\wedge\ (\mathrm{E}(\omega_x) > a_x)\\ &\Longrightarrow\ K = \tfrac{a_x u_x (d^{\mathrm{I}}_y - d_{y|x})(b_{y|x} - b_{y|\overline{x}})}{(d_x + (1-a_x)u_x) a_y (d_{y|\overline{x}} - d_{y|x})}\end{aligned} \tag{4.50}$$

Case II.B.1:

$$\begin{aligned}&((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) > (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))\\ &\wedge\ (\mathrm{E}(\omega_x) \leq a_x)\\ &\Longrightarrow\ K = \tfrac{(1-a_x) u_x (b^{\mathrm{I}}_y - b_{y|\overline{x}})(d_{y|\overline{x}} - d_{y|x})}{(b_x + a_x u_x)(1-a_y)(b_{y|x} - b_{y|\overline{x}})}\end{aligned} \tag{4.51}$$

Case II.B.2:

$$\begin{aligned}&((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) > (b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})))\\ &\wedge\ (\mathrm{E}(\omega_x) > a_x)\\ &\Longrightarrow\ K = \tfrac{(1-a_x) u_x (d^{\mathrm{I}}_y - d_{y|x})}{(d_x + (1-a_x)u_x)(1-a_y)}\end{aligned} \tag{4.52}$$

Case III.A.1:

$$\begin{aligned}&((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) \leq (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))\\ &\wedge\ (\mathrm{E}(\omega_x) \leq a_x)\\ &\Longrightarrow\ K = \tfrac{(1-a_x) u_x (d^{\mathrm{I}}_y - d_{y|\overline{x}})(b_{y|\overline{x}} - b_{y|x})}{(b_x + a_x u_x) a_y (d_{y|x} - d_{y|\overline{x}})}\end{aligned} \tag{4.53}$$

Case III.A.2:

$$\begin{aligned}&((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) \leq (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))\\ &\wedge\ (\mathrm{E}(\omega_x) > a_x)\\ &\Longrightarrow\ K = \tfrac{(1-a_x) u_x (b^{\mathrm{I}}_y - b_{y|x})}{(d_x + (1-a_x)u_x) a_y}\end{aligned} \tag{4.54}$$

Case III.B.1:

$$\begin{aligned}&((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) > (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))\\ &\wedge\ (\mathrm{E}(\omega_x) \leq a_x)\\ &\Longrightarrow\ K = \tfrac{a_x u_x (d^{\mathrm{I}}_y - d_{y|\overline{x}})}{(b_x + a_x u_x)(1-a_y)}\end{aligned} \tag{4.55}$$

Case III.B.2:

$$\begin{aligned}&((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}))\\ &\wedge\ (\mathrm{E}(\omega_{y\|\widehat{x}}) > (b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}})))\\ &\wedge\ (\mathrm{E}(\omega_x) > a_x)\\ &\Longrightarrow\ K = \tfrac{a_x u_x (b^{\mathrm{I}}_y - b_{y|x})(d_{y|x} - d_{y|\overline{x}})}{(d_x + (1-a_x)u_x)(1-a_y)(b_{y|\overline{x}} - b_{y|x})}\end{aligned} \tag{4.56}$$

$$\begin{aligned}\textit{where}\quad &\mathrm{E}(\omega_{y\|\widehat{x}}) &&= b_{y|x}a_x + b_{y|\overline{x}}(1 - a_x) + a_y(u_{y|x}a_x + u_{y|\overline{x}}(1 - a_x))\\ \textit{and}\quad &\mathrm{E}(\omega_x) &&= b_x + a_x u_x\ .\end{aligned} \tag{4.57}$$

Then $\omega_{y\|x}$ is called the conditionally deduced opinion of ω_x by $\omega_{y|x}$ and $\omega_{y|\overline{x}}$. The opinion $\omega_{y\|x}$ expresses the belief in y being true as a function of the beliefs in x and the two sub-conditionals $y|x$ and $y|\overline{x}$. The conditional deduction operator is a ternary operator, and by using the function symbol '$\circledcirc$' to designate this operator, we define $\omega_{y\|x} = \omega_x \circledcirc (\omega_{y|x}, \omega_{y|\overline{x}})$.

Justification for the Binomial Deduction Operator

While not particularly complex, the expressions for conditional inference has many cases which can be difficult to understand and interpret. A more direct and intuitive justification can be found in its geometrical interpretation.

The image space of the consequent opinion is a subtriangle where the two sub-conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ form the two bottom vertices. The third vertex of the subtriangle is the consequent opinion resulting from a vacuous antecedent. This particular consequent opinion, denoted by $\omega_{y\|\widehat{x}}$, is determined by the base rates of x and y as well as the horizontal distance between the sub-conditionals. The antecedent opinion then determines the actual position of the consequent within that subtriangle.

For example, when the antecedent is believed to be TRUE, i.e. $\omega_x = (1, 0, 0, a_x)$, the consequent opinion is $\omega_{y\|x} = \omega_{y|x}$, when the antecedent is believed to be FALSE, i.e. $\omega_x = (0, 1, 0, a_x)$, the consequent opinion is $\omega_{y\|x} = \omega_{y|\overline{x}}$, and when the antecedent opinion is vacuous, i.e. $\omega_x = (0, 0, 1, a_x)$, the consequent opinion is $\omega_{y\|x} = \omega_{y\|\widehat{x}}$. For all other opinion values of the antecedent, the consequent opinion is determined by linear mapping from a point in the antecedent triangle to a point in the consequent subtriangle according to Def.21.

It can be noticed that when $\omega_{y|x} = \omega_{y|\overline{x}}$, the consequent subtriangle is reduced to a point, so that it is necessary that $\omega_{y\|x} = \omega_{y|x} = \omega_{y|\overline{x}} = \omega_{y\|\widehat{x}}$ in this case. This would mean that there is no relevance relationship between antecedent and consequent.

The conditional opinions define a sub-triangle inside the consequent opinion tringle. The space of antecent opinions is then projected to the space of the consequent opinions inside the sub-triangle. Fig.4.7 illustrates the principle of projecting an antecedent opinion to the consequent opinion in the subt-riangle with vertices $\omega_{y|x} = (0.90,\ 0.02,\ 0.08,\ 0.50)$, $\omega_{y|\overline{x}} = (0.40,\ 0.52,\ 0.08,\ 0.50)$ and $\omega_{y\|\widehat{x}} = \omega_{y\|\mathrm{vac}(x)} = (0.40,\ 0.02,\ 0.58,\ 0.50)$.

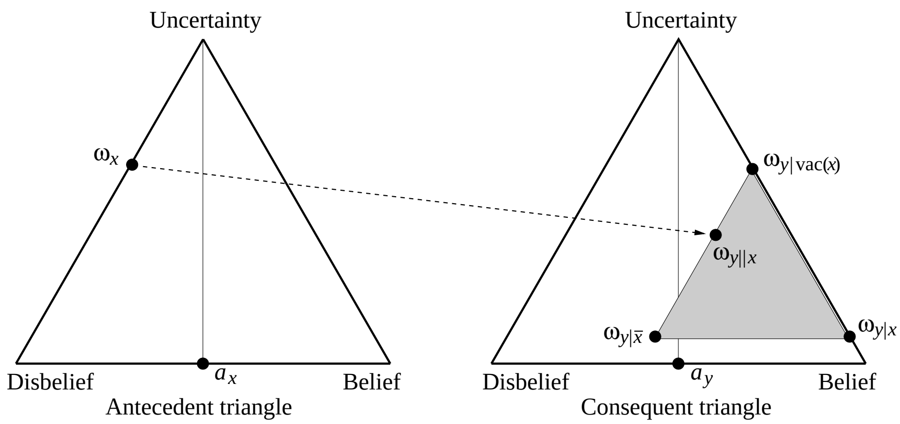


Figure 4.7: Projection from antecedent opinion triangle to the consequent opinion sub-triangle

Let for example the opinion about the antecedent be $\omega_x = (0.00,\ 0.38,\ 0.62,\ 0.50)$. The opinion of the consequent $\omega_{y\|x} = (0.40,\ 0.21,\ 0.39,\ 0.50)$ can then be obtained by mapping the position of the antecedent ω_x in the antecedent triangle onto a position that relatively seen has the same belief, disbelief and uncertainty components in the subtriangle (shaded area) of the consequent triangle.

In the general case, the consequent image subtriangle is not equal sided as in the example above. By setting base rates of x and y different from 0.5, and by defining subconditionals with different uncertainty, the consequent image subtriangle will be skewed, and it is even possible that the uncertainty of $\omega_{y\|\widehat{x}}$ is less that that of $\omega_{x|y}$ or $\omega_{x|\overline{y}}$.

Binomial Abduction

A high level description of conditional abduction with binomial opinions has previously been published in [8]. Abduction requires the conditional opinions to be inverted. Here we will describe the detailed mathematical expressions necessary for computing the required inverted conditionals. Binomial abduction with opinions is a generalisation of probabilistic conditional abduction expressed in Eq.(4.35).

Let x be the parent node, and let y be the child node. Assume that the available conditionals are $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ which are expressed in the opposite direction to what is needed in order to apply Def.21 of conditional deduction. Abduction simply consists of first inverting conditionals to produce $\omega_{y|x}$ and $\omega_{y|\overline{x}}$, and subsequently to apply Def.21.

Deriving the inverted conditional opinions requires knowledge of the base rate a_y of the child proposition y. The process is compatible with, and is based on inverting probabilistic conditionals.

Definition 22 (Conditional Abduction with Binomial Beliefs)

First compute the probability expectation values of the available conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ using Eq.(3.1) to produce:

$$\begin{cases} \mathrm{E}(\omega_{x|y}) & = p(x|y) = b_{x|y} + a_x u_{x|y} \\ \\ \mathrm{E}(\omega_{x|\overline{y}}) & = p(x|\overline{x}) = b_{x|\overline{y}} + a_x u_{x|\overline{y}} \end{cases} \tag{4.58}$$

Following the principle of Eq.(4.36), compute the probability expectation values of the inverted conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ using the values of Eq.(4.58).

$$\begin{cases} \mathrm{E}(\omega_{y|x}) & = p(y|x) = (a_y p(x|y))/(a_y p(x|y) + a_{\overline{y}} p(x|\overline{y})) \\ \\ \mathrm{E}(\omega_{y|\overline{x}}) & = p(y|\overline{x}) = (a_y p(\overline{x}|y))/(a_y p(\overline{x}|y) + a_{\overline{y}} p(\overline{x}|\overline{y})) \end{cases} \tag{4.59}$$

Synthesise a pair of dogmatic conditional opinions from the expectation values of Eq.(4.59):

$$\begin{cases} \underline{\omega}_{y|x} = (p(y|x), p(\overline{y}|x), 0, a_y) \\ \\ \underline{\omega}_{y|\overline{x}} = (p(y|\overline{x}), p(\overline{y}|\overline{x}), 0, a_y) \end{cases} \quad \text{where} \quad p(\overline{y}|x) = (1 - p(y|x)) \quad \text{and} \quad p(\overline{y}|\overline{x}) = (1 - p(y|\overline{x}))\ . \tag{4.60}$$

The expectation values of the dogmatic conditionals of Eq.(4.60) and of the inverted conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ are equal. However, the inverted conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ do in general contain uncertainty, in contrast to the dogmatic opinions of Eq.(4.60) that contain no uncertainty. The inverted conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ can be derived from the dogmatic opinions of Eq.(4.60) by giving them an appropriate amount of uncertainty. This amount of uncertainty is a function of the following elemente:

- the theoretical maximum uncertainty values $u^M_{y|x}$ and $u^M_{y|\overline{x}}$ of $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ respectively,
- the uncertainties $u_{x|y}$ and $u_{x|\overline{y}}$ of the conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ respectively,
- the irrelevance $\overline{\mathrm{R}}(X|Y)$ of the frame Y to the frame X .

The relevance expresses the diagnostic power of the conditionals, i.e. to what degree beliefs in the truth of the antecedents influences beliefs in the truth of the consequents. The relevance $\mathrm{R}(X|Y)$ of the binary frame Y to the frame X is determined by the absolute value of the difference between the expectation values of the conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$, as generally expressed by Eq.(5.5), so that relevance is a value in the range $[0, 1]$. For the purpose of determining uncertainty it is convenient to express the *irrelevance*, expressed as $\overline{\mathrm{R}}(X|Y) = 1 - \textit{relevance}$, or more specifically:

$$\overline{\mathrm{R}}(X|Y) = 1 - \mathrm{R}(X|Y) = 1 - |\mathrm{E}(x|y) - \mathrm{E}(x|\overline{y})|\ . \tag{4.61}$$

The uncertainty of the inverted conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ is an increasing function of the uncertainty of the conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$, because uncertainty in one conditional reasoning direcetion is naturally reflected by uncertainty in the opposite conditional reasoning direcetion. Similarly, the uncertainty of the inverted conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ is an increasing function of the irrelevance $\overline{\mathrm{R}}(X|Y)$, because less relevance is less informative, and thereby lead to more uncertainty. We define the relative uncertainty $u^r_{X|Y}$ to represent the effect of irrelevance and weighted uncertainty.

More specifically, the relative uncertainty $u^r_{X|Y}$ is determined by the disjunctive combination of weighted uncertainty and the irrelevance $\overline{\mathrm{R}}(X|Y)$, meaning that the relative uncertainty $u^r_{X|Y}$ is high in case the weighted uncertainty of $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ is high, or the irrelevance of Y to X is high, or both are high at the same time. The relative uncertainty $u^r_{X|Y}$ is then expressed as:

$$\begin{aligned} u^r_{X|Y} &= a_y(u_{x|y} \sqcup \overline{\mathrm{R}}(X|Y)) + a_{\overline{y}}(u_{x|\overline{y}} \sqcup \overline{\mathrm{R}}(X|Y)) \\ &= a_y(u_{x|y} + \overline{\mathrm{R}}(X|Y) - u_{x|y}\overline{\mathrm{R}}(X|Y)) + a_{\overline{y}}(u_{x|\overline{y}} + \overline{\mathrm{R}}(X|Y) - u_{x|\overline{y}}\overline{\mathrm{R}}(X|Y)). \end{aligned} \tag{4.62}$$

The theoretical maximum uncertainties $u^M_{y|x}$ of $\omega_{y|x}$ and $u^M_{y|\overline{x}}$ of $\omega_{y|\overline{x}}$ are determined by setting either the belief or the disbelief mass to zero according to the simple IF-THEN-ELSE algorithm below. After computing the theoretical maximum uncertainty of each inverted conditional opinion, the uncertainty values of the inverted conditional opinions are computed as the product of the theoretical maximum uncertainty and the relative uncertainty. The other opinion parameters emerge directly.

Computation of $u_{y\|x}$	
IF	$p(y\|x) < a_y$
THEN	$u^M_{y\|x} = p(y\|x)/a_y$
ELSE	$u^M_{y\|x} = (1 - p(y\|x))/(1 - a_y)$

(4.63)

Having computed $u^M_{y|x}$ the opinion parameters can be computed as:

$$\begin{cases} u_{y|x} = u^M_{y|x} u^r_{X|Y} \\ b_{y|x} = p(y|x) - a_y u_{y|x} \\ d_{y|x} = 1 - b_{y|x} - u_{y|x} \end{cases} \quad \text{so that we can express } \omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_y). \tag{4.64}$$

Computation of $u_{y\|\overline{x}}$	
IF	$p(y\|\overline{x}) < a_y$
THEN	$u^M_{y\|\overline{x}} = p(y\|\overline{x})/a_y$
ELSE	$u^M_{y\|\overline{x}} = (1 - p(y\|\overline{x}))/(1 - a_y)$

(4.65)

Having computed $u^M_{y|\overline{x}}$ the opinion parameters can be computed as:

$$\begin{cases} u_{y|\overline{x}} = u^M_{y|\overline{x}} u^r_{X|Y} \\ b_{y|\overline{x}} = p(y|\overline{x}) - a_y u_{y|\overline{x}} \\ d_{y|\overline{x}} = 1 - b_{y|\overline{x}} - u_{y|\overline{x}} \end{cases} \quad \text{so that we can express } \omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_y). \tag{4.66}$$

The inverted conditionals can now be used for conditional deduction according to Def.21. Applying the inverted conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ in deductive reasoning is equivalent to abductive reasoning with the conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$, which can be used to derive the conditionally abduced opinion $\omega_{y\overline{\|}x}$.

The abduction operator, denoted as $\overline{\circledcirc}$, is written as $\omega_{y\overline{\|}x} = \omega_x \overline{\circledcirc}\left(\omega_{x|y},\ \omega_{x|\overline{y}},\ a_y\right)$.

4.6.3 Multinomial Deduction and Abduction with Subjective Opinions

This section presents conditional deduction and abduction with multinomial opinions. These operators are generalisations of binomial operators presented in Sec.4.6.2.

Let $X = \{x_i | i = 1 \ldots k\}$ and $Y = \{y_j | j = 1 \ldots l\}$ be frames, where X will play the role of parent, and Y will play the role of child.

Assume the parent opinion ω_X where $|X| = k$. Assume also the conditional opinions of the form $\omega_{Y|x_i}$, where $i = 1 \ldots k$. There is thus one conditional for each element x_i in the parent frame. Each of these conditionals must be interpreted as the subjective opinion on Y, given that x_i is TRUE. The subscript notation on each conditional opinion indicates not only the frame Y it applies to, but also the element x_i in the antecedent frame it is conditioned on.

By using the notation for probabilistic conditional deduction the corresponding expressions for subjective logic conditional deduction can be defined.

$$\omega_{Y\|X} = \omega_X \circledcirc \omega_{Y|X} \tag{4.67}$$

where $\circledcirc$ is the general conditional deduction operator for subjective opinions, and $\omega_{Y|X} = \{\omega_{Y|x_i} | i = 1 \ldots k\}$ is a set of $k = |X|$ different opinions conditioned on each $x_i \in X$ respectively. Similarly, the expressions for subjective logic conditional abduction can be written as:

$$\omega_{Y\overline{\|}X} = \omega_X \,\overline{\circledcirc}\, (\omega_{X|Y}, a_Y) \tag{4.68}$$

where $\overline{\circledcirc}$ is the general conditional abduction operator for subjective opinions, and $\omega_{X|Y} = \{\omega_{X|y_j} | j = 1 \ldots l\}$ is a set of $l = |Y|$ different multinomial opinions conditioned on each $y_j \in Y$ respectively.

In order to evaluate the expression of Eq.(4.67) and Eq.(4.68), the general deduction and abduction operators must be defined. For binomial opinions, these have been defined in [18, 22]. Deduction and abduction for multinomial opinions are described below [9].

Subjective Logic Deduction

Assume that an observer perceives a conditional relationship between the two frames X and Y. Let $\omega_{Y|X}$ be the set of conditional opinions on the consequent frame Y as a function of the opinion on the antecedent frame X expressed as

$$\omega_{Y|X} : \{\omega_{Y|x_i},\ i = 1, \dots k\}\ . \tag{4.69}$$

Each conditional opinion is a tuple composed of a belief vector $\vec{b}_{Y|x_i}$, an uncertainty mass $u_{Y|x_i}$ and a base rate vector $\vec{a}_Y$ expressed as:

$$\omega_{Y|x_i} = \left(\vec{b}_{Y|x_i}, u_{Y|x_i}, \vec{a}_Y\right)\ . \tag{4.70}$$

Note the base rate vector $\vec{a}_Y$ is equal for all conditional opinions of Eq.(4.69). Let ω_X be the opinion on the antecedent frame X.

Traditional probabilistic conditional deduction can always be derived from these opinions by inserting their probability expectation values into Eq.(4.38), resulting in the expression below.

$$\mathrm{E}(y_j \| X) = \sum_{i=1}^{k} \mathrm{E}(x_i)\mathrm{E}(y_j|x_i) \quad \text{where Eq.(3.7) provides each factor.} \tag{4.71}$$

The operator for subjective logic deduction takes the uncertainty of $\omega_{Y|X}$ and ω_X into account when computing the derived opinion $\omega_{Y\|X}$ as indicated by Eq.(4.67). The method for computing the derived opinion describe below is based on a geometric analysis of the input opinions $\omega_{Y|X}$ and ω_X, and how they relate to each other.

The conditional opinions will in general define a sub-simplex inside the opinion simplex of the consequent child frame Y. The antecedent parent opinion is then linearly projected to the consequent child opiniono inside the sub-simplex. A visualisation of deduction with ternary parent and child tetrahedrons and trinomial opinions is illustrated in Fig.4.8.

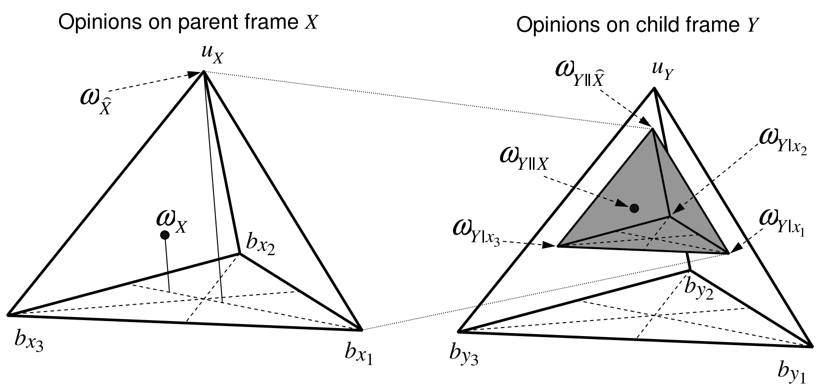


Figure 4.8: Projection from antecedent opinion tetrahedron to consequent opinion sub-tetrahedron.

The sub-simplex formed by the conditional projection of the parent simplex into the child simplex is shown as the shaded tetrahedron on the right hand side in Fig.4.8. The position of the derived opinion $\omega_{Y\|X}$ is geometrically determined by the point inside the sub-simplex that linearly corresponds to the opinion ω_X in the parent simplex.

In general, a sub-simplex will not appear as regular as in the example of Fig.4.8, and can be skewed in all possible ways. The dimensionality of the sub-simplex is equal to the smallest cardinality of X and Y. For binary frames, the sub-simplex is reduced to a triangle. Visualising a simplex larger than ternary is impractical on two-dimensional media such as paper and flat screens.

The mathematical procedure for determining the derived opinion $\omega_{Y\|X}$ is described in four steps below. The uncertainty of the sub-simplex apex will emerge from the largest sub-triangle in any dimension of Y when projected against the triangular side planes, and is derived in steps 1 to 3 below. The following expressions are needed for the computations.

$$\begin{cases} \mathrm{E}(y_t|\widehat{X}) & = \sum_{i=1}^{k} a_{x_i}\mathrm{E}(y_t|x_i)\ , \\ \\ \mathrm{E}(y_t|(\widehat{x_r, x_s})) & = (1 - a_{y_t})b_{y_t|x_s} + a_{y_t}(b_{y_t|x_r} + u_{Y|x_r})\ . \end{cases} \tag{4.72}$$

The expression $\mathrm{E}(y_t|\widehat{X})$ gives the expectation value of y_t given a vacuous opinion $\omega_{\widehat{X}}$ on X. The expression $\mathrm{E}(y_t|(\widehat{x_r, x_s}))$ gives the expectation value of y_t for the theoretical maximum uncertainty $u^{\mathrm{T}}_{y_t}$.

- **Step 1:** Determine the X-dimensions (x_r, x_s) that give the largest theoretical uncertainty $u^{\mathrm{T}}_{y_t}$ in each Y-dimension y_t, independently of the opinion on X. Each dimension's maximum uncertainty is:

$$u^{\mathrm{T}}_{y_t} = 1 - \mathrm{Min}\Big[\Big(1 - b_{y_t|x_r} - u_{Y|x_r} + b_{y_t|x_s}\Big), \forall (x_r, x_s) \in X\Big] \;. \tag{4.73}$$

The X-dimensions (x_r, x_s) are recorded for each y_t. Note that it is possible to have $x_r = x_x$.

- **Step 2:** First determine the triangle apex uncertainty $u_{y_t\|\widehat{X}}$ for each Y-dimension by assuming a vacuous opinion $\omega_{\widehat{X}}$ and the actual base rate vector $\vec{a}_X$. Assuming that $a_{y_t} \neq 0$ and $a_{y_t} \neq 1$ for all base rates on Y, each triangle apex uncertainty $u_{y_t\|\widehat{X}}$ can be computed as:

Case A: $\mathrm{E}(y_t|\widehat{X}) \leq \mathrm{E}(y_t|(\widehat{x_r, x_s}))$:

$$u_{y_t\|\widehat{X}} = \left(\frac{\mathrm{E}(y_t|\widehat{X}) - b_{y_t|x_s}}{a_{y_t}}\right) \tag{4.74}$$

Case B: $\mathrm{E}(y_t|\widehat{X}) > \mathrm{E}(y_t|(\widehat{x_r, x_s}))$:

$$u_{y_t\|\widehat{X}} = \left(\frac{b_{y_t|x_r} + u_{Y|x_r} - \mathrm{E}(y_t|\widehat{X})}{1 - a_{y_t}}\right) \tag{4.75}$$

Then determine the intermediate sub-simplex apex uncertainty $u^{\mathrm{Int}}_{Y\|\widehat{X}}$ which is equal to the largest of the triangle apex uncertainties computed above. This uncertainty is expressed as.

$$u^{\mathrm{Int}}_{Y\|\widehat{X}} = \mathrm{Max}\left[u_{y_t\|\widehat{X}}, \forall y_t \in Y\right] \;. \tag{4.76}$$

- **Step 3:** First determine the intermediate belief components $b^{\mathrm{Int}}_{y_j\|\widehat{X}}$ in case of vacuous belief on X as a function of the intermediate apex uncertainty $u^{\mathrm{Int}}_{Y\|\widehat{X}}$:

$$b^{\mathrm{Int}}_{y_j\|\widehat{X}} = \mathrm{E}(y_j\|\widehat{X}) - a_{y_j} u^{\mathrm{Int}}_{Y\|\widehat{X}} \tag{4.77}$$

For particular geometric combinations of the triangle apex uncertainties $u_{y_t\|\widehat{X}}$ it is possible that an intermediate belief component $b^{\mathrm{Int}}_{y_j\|\widehat{X}}$ becomes negative. In such cases a new adjusted apex uncertainty $u^{\mathrm{Adj}}_{y_t\|\widehat{X}}$ is computed. Otherwise the adjusted apex uncertainty is set equal to the intermediate apex uncertainty of Eq.(4.76). Thus:

Case A: $b^{\mathrm{Int}}_{y_j\|\widehat{X}} < 0$: $\quad u^{\mathrm{Adj}}_{y_j\|\widehat{X}} = \mathrm{E}(y_j\|\widehat{X})/a_{y_j}$ (4.78)

Case B: $b^{\mathrm{Int}}_{y_j\|\widehat{X}} \geq 0$: $\quad u^{\mathrm{Adj}}_{y_j\|\widehat{X}} = u^{\mathrm{Int}}_{Y\|\widehat{X}}$ (4.79)

Then compute the sub-simplex apex uncertainty $u_{Y\|\widehat{X}}$ as the minimum of the adjusted apex uncertainties according to:

$$u_{Y\|\widehat{X}} = \mathrm{Min}\left[u^{\mathrm{Adj}}_{y_j\|\widehat{X}}, \forall y_t \in Y\right] \;. \tag{4.80}$$

Note that the apex uncertainty is not necessarily the highest uncertainty of the sub-simplex. It is possible that one of the conditionals $\omega_{Y|x_i}$ actually contains a higher uncertainty, which would simply mean that the sub-simplex is skewed or tilted to the side.

- **Step 4:** Based on the sub-simplex apex uncertainty $u_{Y\|\widehat{X}}$, the actual uncertainty $u_{Y\|X}$ as a function of the opinion on X is:

$$u_{Y\|X} = u_{Y\|\widehat{X}} - \sum_{i=1}^{k}(u_{Y\|\widehat{X}} - u_{Y|x_i})b_{x_i} \ . \tag{4.81}$$

Given the actual uncertainty $u_{Y\|X}$, the actual beliefs $b_{y_j\|X}$ are:

$$b_{y_j\|X} = \mathrm{E}(y_j\|X) - a_{y_j}u_{Y\|X} \ . \tag{4.82}$$

The belief vector $\vec{b}_{Y\|X}$ is expressed as:

$$\vec{b}_{Y\|X} = \left\{b_{y_j\|X} \mid j = 1, \dots l\right\} \ . \tag{4.83}$$

Finally, the derived opinion $\omega_{Y\|X}$ is the tuple composed of the belief vector of Eq.(4.83), the uncertainty belief mass of Eq.(4.81) and the base rate vector of Eq.(4.70) expressed as:

$$\omega_{Y\|X} = \left(\vec{b}_{Y\|X}, u_{Y\|X}, \vec{a}_Y\right) \ . \tag{4.84}$$

The method for multinomial deduction described above represents both a simplification and a generalisation of the method for binomial deduction described in [18]. In case of 2×2 deduction in particular, the methods are equivalent and produce exactly the same results.

Subjective Logic Abduction

Subjective logic abduction requires the inversion of conditional opinions of the form $\omega_{X|y_j}$ into conditional opinions of the form $\omega_{Y|x_i}$ similarly to Eq.(4.39). The inversion of probabilistic conditionals according to Eq.(4.39) uses the division operator for probabilities. While a division operator for binomial opinions is defined in [13], a division operator for multinomial opinions would be complex because it involves matrix and vector expressions. Instead we define inverted conditional opinions as an uncertainty maximised opinion.

It is natural to define base rate opinions as vacuous opinions, so that the base rate vector $\vec{a}$ alone defines their probability expectation values. The rationale for defining inversion of conditional opinions as producing maximum uncertainty is that it involves multiplication with a vacuous base rate opinion which produces an uncertainty maximised product. Let $|X| = k$ and $|Y| = l$, and assume the set of available conditionals to be:

$$\omega_{X|Y} : \left\{\omega_{X,y_j}, \text{ where } j = 1 \dots l\right\} \ . \tag{4.85}$$

Assume further that the analyst requires the set of conditionals expressed as:

$$\omega_{Y|X} : \{\omega_{Y,x_i}, \text{ where } i = 1 \dots k\} \ . \tag{4.86}$$

First compute the l different probability expectation values of each inverted conditional opinion $\omega_{Y|x_i}$, according to Eq.(4.39) as:

$$\mathrm{E}(y_j|x_i) = \frac{a(y_j)\mathrm{E}(\omega_{X|y_j}(x_i))}{\sum_{t=1}^{l} a(y_t)\mathrm{E}(\omega_{X|y_t}(x_i))} \tag{4.87}$$

where $a(y_j)$ denotes the base rate of y_j. Consistency requires that:

$$\mathrm{E}(\omega_{Y|x_i}(y_j)) = \mathrm{E}(y_j|x_i) \ . \tag{4.88}$$

The simplest opinions that satisfy Eq.(4.88) are the k dogmatic opinions:

$$\underline{\omega}_{Y|x_i} : \begin{cases} b_{Y|x_i}(y_j) & = \mathrm{E}(y_j|x_i), \quad \text{for } j = 1 \dots k \ , \\ u_{Y|x_i} & = 0 \ , \\ \vec{a}_{Y|x_i} & = \vec{a}_Y \ . \end{cases} \tag{4.89}$$

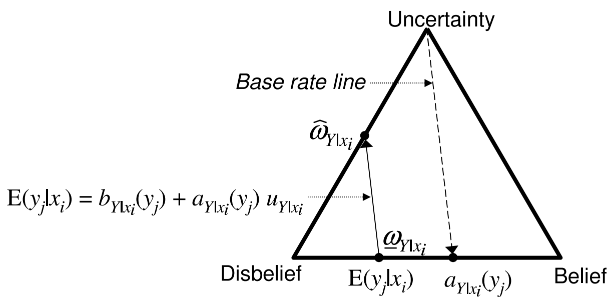


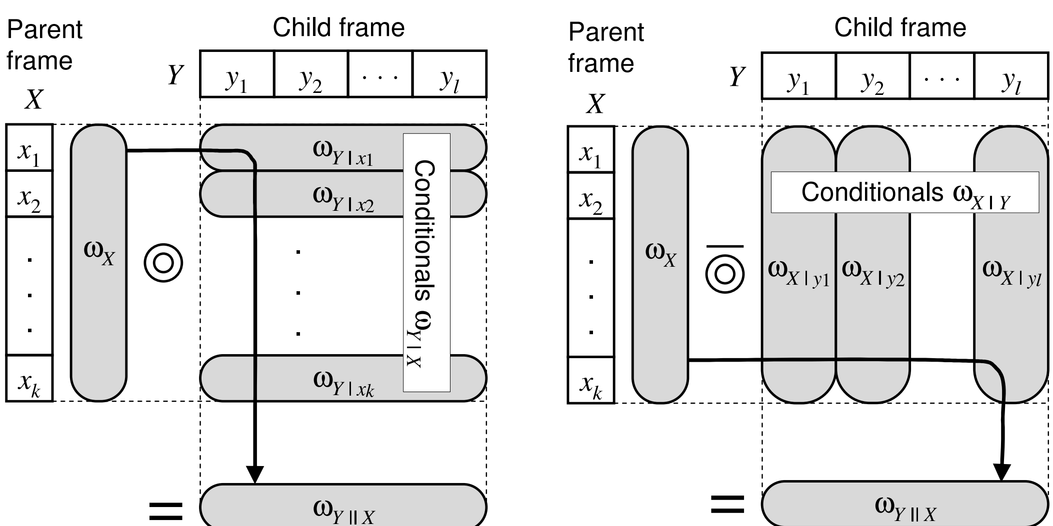
Figure 4.9: Uncertainty maximisation of dogmatic opinion

Uncertainty maximisation of $\underline{\omega}_{Y|x_i}$ consists of converting as much belief mass as possible into uncertainty mass while preserving consistent probability expectation values according to Eq.(4.88). The result is the uncertainty maximised opinion denoted as $\widehat{\omega}_{Y|x_i}$. This process is illustrated in Fig.4.9.

It must be noted that Fig.4.9 only represents two dimensions of the multinomial opinions on Y, namely y_j and its complement. The line defined by

$$\mathrm{E}(y_j|x_i) = b_{Y|x_i}(y_j) + a_{Y|x_i}(y_j)u_{Y|x_i} \ . \tag{4.90}$$

that is parallel to the base rate line and that joins $\underline{\omega}_{Y|x_i}$ and $\widehat{\omega}_{Y|x_i}$ in Fig.4.9, defines the opinions $\omega_{Y|x_i}$ for which the probability expectation values are consistent with Eq.(4.88). A opinion $\widehat{\omega}_{Y|x_i}$ is uncertainty maximised when Eq.(4.90) is satisfied and at least one belief mass of $\widehat{\omega}_{Y|x_i}$ is zero. In general, not all belief masses can be zero simultaneously except for vacuous opinions.



(a) Deduction. (b) Abduction.

Figure 4.10: Visualising deduction and abduction with opinions

In order to find the dimension(s) that can have zero belief mass, the belief mass will be set to zero in Eq.(4.90) successively for each dimension $y_j \in Y$, resulting in l different uncertainty values defined as:

$$u^j_{Y|x_i} = \frac{\mathrm{E}(y_j|x_i)}{a_{Y|x_i}(y_j)}, \text{ where } j = 1 \dots l \ . \tag{4.91}$$

The minimum uncertainty expressed as $\mathrm{Min}\{u^j_{Y|x_i}, \text{ for } j = 1 \dots l\}$ determines the dimension which will have zero belief mass. Setting the belief mass to zero for any other dimension would result in negative belief mass for other dimensions. Assume that y_t is the dimension for which the uncertainty is minimum. The uncertainty maximised opinion can then be determined as:

$$\widehat{\omega}_{Y|x_i} : \begin{cases} b_{Y|x_i}(y_j) &= \mathrm{E}(y_j|x_i) - a_Y(y_j)u^t_{Y|x_i}, \text{ for } y = 1 \dots l \\ u_{Y|x_i} &= u^t_{Y|x_i} \\ \vec{a}_{Y|x_i} &= \vec{a}_Y \end{cases} \quad (4.92)$$

By defining $\omega_{Y|x_i} = \widehat{\omega}_{Y|x_i}$, the expressions for the set of inverted conditional opinions $\omega_{Y|x_i}$ (with $i = 1 \dots k$) emerges. Conditional abduction according to Eq.(4.68) with the original set of conditionals $\omega_{X|Y}$ is now equivalent to conditional deduction according to Eq.(4.67) as described above, where the set of inverted conditionals $\omega_{Y|X}$ is used deductively.

Example: Military Intelligence Analysis with Subjective Logic

The difference between deductive and abductive reasoning with opinions is illustrated in Fig.4.10 below.

Table 4.6: Conditional opinion $\omega_{X|Y}$: troop movement x_i given military plan y_j

Opinions $\omega_{X\|Y}$	Troop movements x_1 : No movemt.	x_2 : Minor movemt.	x_3 : Full mob.	X Any
$\omega_{X\|y_1}$:	$b(x_1) = 0.50$	$b(x_2) = 0.25$	$b(x_3) = 0.25$	$u = 0.00$
$\omega_{X\|y_2}$:	$b(x_1) = 0.00$	$b(x_2) = 0.50$	$b(x_3) = 0.50$	$u = 0.00$
$\omega_{X\|y_3}$:	$b(x_1) = 0.00$	$b(x_2) = 0.25$	$b(x_3) = 0.75$	$u = 0.00$

Table 4.7: Conditional opinions $\omega_{Y|X}$: military plan y_j given troop movement x_i

Military plan	Opinions of military plans given troop movement: $\omega_{Y\|x_1}$ No movemt.	$\omega_{Y\|x_2}$ Minor movemt.	$\omega_{Y\|x_3}$ Full mob.
y_1: No aggression	$b(y_1) = 1.00$	$b(y_1) = 0.00$	$b(y_1) = 0.00$
y_2: Minor ops.	$b(y_2) = 0.00$	$b(y_2) = 0.17$	$b(y_2) = 0.14$
y_3: Invasion	$b(y_3) = 0.00$	$b(y_3) = 0.00$	$b(y_3) = 0.14$
Y: Any	$u = 0.00$	$u = 0.83$	$u = 0.72$

Fig.4.10 shows that deduction uses conditionals defined over the child frame, and that abduction uses conditionals defined over the parent frame.

In this example we revisit the intelligence analysis situation of Sec.4.6.1, but now with conditionals and evidence represented as subjective opinions according to Table 4.6 and Eq.(4.93).

The opinion about troop movements is expressed as the opinion:

$$\omega_X = \begin{cases} b(x_1) &= 0.00, & a(x_1) &= 0.70 \\ b(x_2) &= 0.50, & a(x_2) &= 0.20 \\ b(x_3) &= 0.50, & a(x_3) &= 0.10 \\ u &= 0.00 \end{cases} \quad (4.93)$$

First the conditional opinions must be inverted as expressed in Table 4.7.

Then the likelihoods of country A's plans can be computed as the opinion:

$$\omega_{Y\overline{\|}X} = \begin{cases} b(y_1) &= 0.00, & a(y_1) &= 0.70, & \mathrm{E}(y_1) &= 0.54 \\ b(y_2) &= 0.16, & a(y_2) &= 0.20, & \mathrm{E}(y_2) &= 0.31 \\ b(y_3) &= 0.07, & a(y_3) &= 0.10, & \mathrm{E}(y_3) &= 0.15 \\ u &= 0.77 \end{cases} \quad (4.94)$$

These results can be compared with those of Eq.(4.45) which were derived with probabilities only, and which are equal to the probability expectation values given in the rightmost column of Eq.(4.94). The important observation to make is that although y_1 (no aggression) seems to be country A's most likely plan in probabilistic terms, this likelihood is based on uncertainty only. The only firm evidence actually supports y_2 (minor aggression) or y_3 (full invasion), where y_2 has the strongest support ($b(y_2) = 0.16$). A likelihood expressed as a scalar probability can thus hide important aspects of the analysis, which will only come to light when uncertainty is explicitly expressed, as done in the example above.

4.7 Fusion of Multinomial Opinions

In many situations there will be multiple sources of evidence, and fusion can be used to combine evidence from different sources.

In order to provide an interpretation of fusion in subjective logic it is useful to consider a process that is observed by two sensors. A distinction can be made between two cases.

1. The two sensors observe the process during disjoint time periods. In this case the observations are independent, and it is natural to simply add the observations from the two sensors, and the resulting fusion is called *cumulative fusion*.

2. The two sensors observe the process during the same time period. In this case the observations are dependent, and it is natural to take the average of the observations by the two sensors, and the resulting fusion is called *averaging fusion*.

Fusion of binomial opinions have been described in [6, 7]. The two types of fusion for multinomial opinions are described in the following sections. When observations are partially dependent, a hybrid fusion operator can be defined [15].

4.7.1 The Cumulative Fusion Operator

The cumulative fusion rule is equivalent to *a posteriori* updating of Dirichlet distributions. Its derivation is based on the bijective mapping between the belief and evidence notations described in Eq.(3.20).

Assume a frame X containing k elements. Assume two observers A and B who observe the outcomes of the process over two separate time periods.

Let the two observers' respective observations be expressed as $\vec{r}^A$ and $\vec{r}^B$. The evidence opinions resulting from these separate bodies of evidence can be expressed as $(\vec{r}^A, \vec{a})$ and $(\vec{r}^B, \vec{a})$

The cumulative fusion of these two bodies of evidence simply consists of vector addition of $\vec{r}^A$ and $\vec{r}^B$, expressed as:

$$(\vec{r}^A, \vec{a}) \oplus (\vec{r}^B, \vec{a}) = ((\vec{r}^A + \vec{r}^B), \vec{a})\ . \tag{4.95}$$

The symbol "$\diamond$" denotes the fusion of two observers A and B into a single imaginary observer denoted as $A \diamond B$. All the necessary elements are now in place for presenting the cumulative rule for belief fusion.

Theorem 3 Cumulative Fusion Rule

Let ω^A and ω^B be opinions respectively held by agents A and B over the same frame $X = \{x_i \mid i = 1, \cdots, l\}$. Let $\omega^{A\diamond B}$ be the opinion such that:

Case I: For $u^A \neq 0 \ \vee \ u^B \neq 0$:

$$\begin{cases} b^{A\diamond B}(x_i) = \frac{b^A(x_i)u^B + b^B(x_i)u^A}{u^A + u^B - u^A u^B} \\ u^{A\diamond B} = \frac{u^A u^B}{u^A + u^B - u^A u^B} \end{cases} \tag{4.96}$$

Case II: For $u^A = 0 \ \wedge \ u^B = 0$:

$$\begin{cases} b^{A\diamond B}(x_i) = \gamma^A\, b^A(x_i) + \gamma^B b^B(x_i) \\ u^{A\diamond B} = 0 \end{cases} \quad \text{where} \quad \begin{cases} \gamma^A = \lim\limits_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^B}{u^A + u^B} \\ \gamma^B = \lim\limits_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^A}{u^A + u^B} \end{cases} \tag{4.97}$$

Then $\omega^{A\diamond B}$ is called the cumulatively fused bba of ω^A and ω^B, representing the combination of independent opinions of A and B. By using the symbol '$\oplus$' to designate this belief operator, we define $\omega^{A\diamond B} \equiv \omega^A \oplus \omega^B$.

It can be verified that the cumulative rule is commutative, associative and non-idempotent. In Case II of Theorem 3, the associativity depends on the preservation of relative weights of intermediate results, which requires the additional weight variable γ. In this case, the cumulative rule is equivalent to the weighted average of probabilities.

It is interesting to notice that the expression for the cumulative rule is independent of the *a priori* constant C. That means that the choice of a uniform Dirichlet distribution in the binary case in fact only influences the mapping between Dirichlet distributions and Dirichlet bbas, not the cumulative rule itself. This shows that the cumulative rule is firmly based on classical statistical analysis, and not dependent on arbitrary and ad hoc choices.

The cumulative rule represents a generalisation of the consensus operator [7, 6] which emerges directly from Theorem 3 by assuming a binary frame.

4.7.2 The Averaging Fusion Operator

The average rule is equivalent to averaging the evidence of Dirichlet distributions. Its derivation is based on the bijective mapping between the belief and evidence notations of Eq.(3.20).

Assume a frame X containing k elements. Assume two observers A and B who observe the outcomes of the process over the same time periods.

Let the two observers' respective observations be expressed as $\vec{r}^A$ and $\vec{r}^B$. The evidence opinions resulting from these separate bodies of evidence can be expressed as $(\vec{r}^A, \vec{a})$ and $(\vec{r}^B, \vec{a})$

The averaging fusion of these two bodies of evidence simply consists of averaging $\vec{r}^A$ and $\vec{r}^B$. In terms of Dirichlet distributions, this can be expressed as:

$$(\vec{r}^A, \vec{a})\underline{\oplus}(\vec{r}^B, \vec{a}) = ((\frac{\vec{r}^A + \vec{r}^B}{2}), \vec{a}) \ . \tag{4.98}$$

The symbol "$\underline{\diamond}$" denotes the averaging fusion of two observers A and B into a single imaginary observer denoted as $A \diamond B$.

Theorem 4 Averaging Fusion Rule

Let ω^A and ω^B be opinions respectively held by agents A and B over the same frame $X = \{x_i \mid i = 1, \cdots, l\}$. Let $\omega^{A\underline{\diamond}B}$ be the opinion such that:

Case I: For $u^A \neq 0 \ \vee \ u^B \neq 0$:

$$\begin{cases} b^{A\underline{\diamond}B}(x_i) = \frac{b^A(x_i)u^B + b^B(x_i)u^A}{u^A+u^B} \\ u^{A\underline{\diamond}B} = \frac{2u^A u^B}{u^A+u^B} \end{cases} \tag{4.99}$$

Case II: For $u^A = 0 \ \wedge \ u^B = 0$:

$$\begin{cases} b^{A\diamond B}(x_i) = \gamma^A\, b^A(x_i) + \gamma^B b^B(x_i) \\ u^{A\diamond B} = 0 \end{cases} \quad \textit{where} \quad \begin{cases} \gamma^A = \lim\limits_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^B}{u^A+u^B} \\ \gamma^B = \lim\limits_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^A}{u^A+u^B} \end{cases} \tag{4.100}$$

Then $\omega^{A\underline{\diamond}B}$ is called the averaged opinion of ω^A and ω^B, representing the combination of the dependent opinions of A and B. By using the symbol '$\underline{\oplus}$' to designate this belief operator, we define $\omega^{A\underline{\diamond}B} \equiv \omega^A\underline{\oplus}\omega^B$.

It can be verified that the averaging fusion rule is commutative and idempotent, but not associative.

The cumulative rule represents a generalisation of the consensus rule for dependent opinions defined in [12].

4.8 Trust Transitivity

Assume two agents A and B where A trusts B, and B believes that proposition x is true. Then by transitivity, agent A will also believe that proposition x is true. This assumes that B recommends x to A. In our approach, trust and belief are formally expressed as opinions. The transitive linking of these two opinions consists of discounting B's opinion about x by A's opinion about B, in order to derive A's opinion about x. This principle is illustrated in Fig.4.11 below. The solid arrows represent initial direct trust, and the dotted arrow represents derived indirect trust.

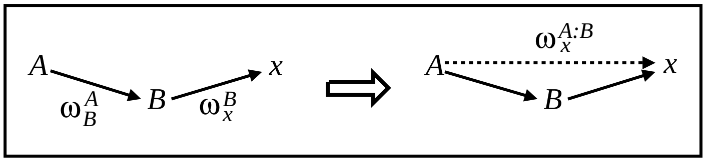


Figure 4.11: Principle of the discounting operator

Trust transitivity, as trust itself, is a human mental phenomenon, so there is no such thing as objective transitivity, and trust transitivity therefore lends itself to different interpretations. We see two main difficulties. The first is related to the effect of A disbelieving that B will give a good advice. What does this exactly mean? We will give two different interpretations and definitions. The second difficulty relates to the effect of base rate trust in a transitive path. We will briefly examine this, and provide the definition of a base rate sensitive discounting operator as an alternative to the two previous which are base rate insensitive.

4.8.1 Uncertainty Favouring Trust Transitivity

A's disbelief in the recommending agent B means that A thinks that B ignores the truth value of x. As a result A also ignores the truth value of x.

Definition 23 (Uncertainty Favouring Discounting) *Let A, B and be two agents where A's opinion about B's recommendations is expressed as $\omega^A_B = \{b^A_B, d^A_B, u^A_B, a^A_B\}$, and let x be a proposition where B's opinion about x is recommended to A with the opinion $\omega^B_x = \{b^B_x, d^B_x, u^B_x, a^B_x\}$. Let $\omega^{A:B}_x = \{b^{A:B}_x, d^{A:B}_x, u^{A:B}_x, a^{A:B}_x\}$ be the opinion such that:*

$$\begin{cases} b^{A:B}_x = b^A_B b^B_x \\ d^{A:B}_x = b^A_B d^B_x \\ u^{A:B}_x = d^A_B + u^A_B + b^A_B u^B_x \\ a^{A:B}_x = a^B_x \end{cases}$$

then $\omega^{A:B}_x$ is called the uncertainty favouring discounted opinion of A. By using the symbol $\otimes$ to designate this operation, we get $\omega^{A:B}_x = \omega^A_B \otimes \omega^B_x$. □

It is easy to prove that this operator is associative but not commutative. This means that the combination of opinions can start in either end of the path, and that the order in which opinions are combined is significant. In a path with more than one recommending entity, opinion independence must be assumed, which for example translates into not allowing the same entity to appear more than once in a transitive path. Fig.4.12 illustrates an example of applying the discounting operator for independent opinions, where $\omega^A_B = \{0.1, 0.8, 0.1\}$ discounts $\omega^B_x = \{0.8, 0.1, 0.1\}$ to produce $\omega^{A:B}_x = \{0.08, 0.01, 0.91\}$.

4.8.2 Opposite Belief Favouring

A's disbelief in the recommending agent B means that A thinks that B consistently recommends the opposite of his real opinion about the truth value of x. As a result, A not only disbelieves in x to the degree that B recommends belief, but she also believes in x to the degree that B recommends disbelief in x, because the combination of two disbeliefs results in belief in this case.

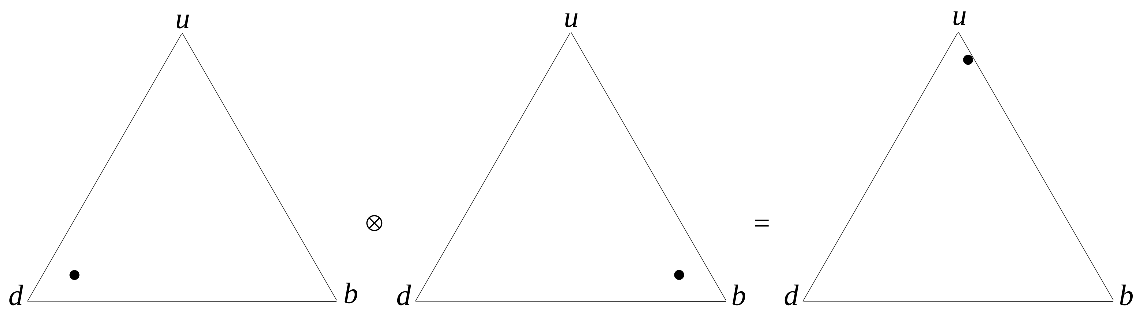


Figure 4.12: Example of applying the discounting operator for independent opinions

Definition 24 (Opposite Belief Favouring Discounting) *Let A, B and be two agents where A's opinion about B's recommendations is expressed as* $\omega^A_B = \{b^A_B, d^A_B, u^A_B, a^A_B\}$, *and let x be a proposition where B's opinion about x is recommended to A as the opinion* $\omega^B_x = \{b^B_x, d^B_x, u^B_x, a^B_x\}$. *Let* $\omega^{A:B}_x = \{b^{A:B}_x, d^{A:B}_x, u^{A:B}_x, a^{A:B}_x\}$ *be the opinion such that:*

$$\begin{cases} b^{A:B}_x = b^A_B b^B_x + d^A_B d^B_x \\ d^{A:B}_x = b^A_B d^B_x + d^A_B b^B_x \\ u^{A:B}_x = u^A_B + (b^A_B + d^A_B)u^B_x \\ a^{A:B}_x = a^B_x \end{cases}$$

then $\omega^{A:B}_x$ *is called the opposite belief favouring discounted recommendation from B to A. By using the symbol* $\otimes$ *to designate this operation, we get* $\omega^{A:B}_x = \omega^A_B \otimes \omega^B_x$. □

This operator models the principle that *"your enemy's enemy is your friend"*. That might be the case in some situations, and the operator should only be applied when the situation makes it plausible. It is doubtful whether it is meaningful to model more than two arcs in a transitive path with this principle. In other words, it is doubtful whether the enemy of your enemy's enemy necessarily is your enemy too.

4.8.3 Base Rate Sensitive Transitivity

In the transitivity operators defined in Sec.4.8.1 and Sec.4.8.2 above, a^A_B had no influence on the discounting of of the recommended (b^B_x, d^B_x, u^B_x) parameters. This can seem counterintuitive in many cases such as in the example described next.

Imagine a stranger coming to a town which is know for its citizens being honest. The stranger is looking for a car mechanic, and asks the first person he meets to direct him to a good car mechanic. The stranger receives the reply that there are two car mechanics in town, David and Eric, where David is cheap but does not always do quality work, and Eric might be a bit more expensive, but he always does a perfect job.

Translated into the formalism of subjective logic, the stranger has no other info about the person he asks than the base rate that the citizens in the town are honest. The stranger is thus ignorant, but the expectation value of a good advice is still very high. Without taking a^A_B into account, the result of the definitions above would be that the stranger is completely ignorant about which if the mechanics is the best.

An intuitive approach would then be to let the expectation value of the stranger's trust in the recommender be the discounting factor for the recommended (b^B_x, d^B_x) parameters.

Definition 25 (Base Rate Sensitive Discounting) *The base rate sensitive discounting of a belief* $\omega^B_x = (b^B_x, d^B_x, u^B_x, a^B_x)$ *by a belief* $\omega^A_B = (b^A_B, d^A_B, u^A_B, a^A_B)$ *produces the transitive belief* $\omega^{A:B}_x = (b^{A:B}_x, d^{A:B}_x, u^{A:B}_x, a^{A:B}_x)$ *where*

$$\begin{cases} b^{A:B}_x = & \mathrm{E}(\omega^A_B) b^B_x \\ d^{A:B}_x = & \mathrm{E}(\omega^A_B) d^B_x \\ u^{A:B}_x = & 1 - \mathrm{E}(\omega^A_B)(b^B_x + d^B_x) \\ a^{A:B}_x = & a^B_x \end{cases} \tag{4.101}$$

where the probability expectation value $\mathrm{E}(\omega^A_B) = b^A_B + a^A_B u^A_B$.

However this operator must be applied with care. Assume again the town of honest citizens, and let let the stranger A have the opinion $\omega^A_B = (0,\ 0,\ 1,\ 0.99)$ about the first person B she meets, i.e. the opinion has no basis in evidence other than a very high base rate defined by $a^A_B = 0.99$. If the person B now recommends to A the opinion $\omega^B_x = (1,\ 0,\ 0,\ a)$, then, according to the base rate sensitive discounting operator of Def.25, A will have the belief $\omega^{A:B}_x = (0.99,\ 0,\ 0.01,\ a)$ in x. In other words, the highly certain belief $\omega^{A:B}_x$ is derived on the basis of the highly uncertain belief ω^A_B, which can seem counterintuitive. This potential problem could be amplified as the trust path gets longer. A safety principle could therefore be to only apply the base rate sensitive discounting to the last transitive link.

There might be other principles that better reflect human intuition for trust transitivity, but we will leave this question to future research. It would be fair to say that the base rate insensitive discounting operator of Def.23 is safe and conservative, and that the base rate sensitive discounting operator of Def.25 can be more intuitive in some situations, but must be applied with care.

4.9 Belief Constraining

Situations where agents with different preferences try to agree on a single choice occur frequently. This must not be confused with fusion of evidence from different agents to determine the most likely correct hypothesis or actual event. Multi-agent preference combination assumes that each agent has already made up her mind, and is about determining the most acceptable decision or choice for the group of agents. Preferences for a state variable can be expressed in the form of subjective opinions over a frame. The belief constraint operator of subjective logic can be applied as a method for merging preferences of multiple agents into a single preference for the whole group. This model is expressive and flexible, and produces perfectly intuitive results. Preference can be represented as belief and indifference can be represented as uncertainty/uncommitted belief. Positive and negative preferences are considered as symmetric concepts, so they can be represented in the same way and combined using the same operator. A totally uncertain opinion has no influence and thereby represents the neutral element.

The belief constraint operator [16, 17] described here is an extension of Dempster's rule which in Dempster-Shafer belief theory is often presented as a method for fusing evidence from different sources [24]. Many authors have however demonstrated that Dempster's rule is not an appropriate operator for evidence fusion [26], and that it is better suited as a method for combining constraints [14], which is also our view.

4.9.1 The Belief Constraint Operator

Assume two opinions ω^A_X and ω^B_X over the frame X. The superscripts A and B are attributes that identify the respective belief sources or belief owners. These two opinions can be mathematically merged using the belief constraint operator denoted as "$\odot$" which in formulas is written as: $\omega^{A\&B}_X = \omega^A_X \odot \omega^B_X$. Belief source combination denoted with "&" thus represents opinion combination with "$\odot$". The algebraic expression of the belief constraint operator for subjective opinions is defined next.

Definition 26 (Belief Constraint Operator)

$$\omega^{A\&B}_X = \omega^A_X \odot \omega^B_X = \begin{cases} \vec{b}^{A\&B}(x_i) = \frac{Har(x_i)}{(1-Con)}, & \forall\ x_i \in \mathcal{R}(X),\ x_i \neq \emptyset \\ \\ u^{A\&B}_X = \frac{u^A_X u^B_X}{(1-Con)} & \\ \\ \vec{a}^{A\&B}(x_i) = \frac{\vec{a}^A(x_i)(1-u^A_X)+\vec{a}^B(x_i)(1-u^B_X)}{2-u^A_X-u^B_X}, & \forall\ x_i \in X,\ x_i \neq \emptyset \end{cases} \tag{4.102}$$

The term $Har(x_i)$ represents the degree of *Harmony*, or in other words overlapping belief mass, on x_i. The term *Con* represents the degree of belief *Conflict*, or in other words non-overlapping belief mass, between ω^A_X and ω^B_X.

These are defined below:

$$\begin{aligned} Har(x_i) &= \vec{b}^A(x_i)u^B_X + \vec{b}^B(x_i)u^A_X + \sum_{y\cap z=x_i} \vec{b}^A(y)\vec{b}^B(z), \qquad \forall\ x_i \in \mathcal{R}(X)\ . \\ Con &= \sum_{y\cap z=\emptyset} \vec{b}^A(y)\vec{b}^B(z)\ . \end{aligned} \tag{4.103}$$

The purpose of the divisor $(1 - Con)$ in Eq.(4.102) is to normalise the derived belief mass, or in other words to ensure belief mass and uncertainty mass additivity. The use of the belief constraint operator is mathematically possible only if ω^A and ω^B are not totally conflicting, i.e., if $Con \neq 1$.

The belief constraint operator is commutative and non-idempotent. Associativity is preserved when the base rate is equal for all agents. Associativity in case of different base rates requires that all preference opinions be combined in a single operation which would require a generalisation of Def.26 for multiple agents, i.e. for multiple input arguments, which is relatively trivial. A totally indifferent opinion acts as the neutral element for belief constraint, formally expressed as:

$$\text{IF } (\omega^A_X \text{ is totally indifferent, i.e. with } u^A_X = 1) \text{ THEN } (\omega^A_X \odot \omega^B_X = \omega^B_X)\ . \tag{4.104}$$

Having a neutral element in the form of the totally indifferent opinion is very useful when modelling situations of preference combination.

The flexibility of subjective logic makes it simple to express positive and negative preferences within the same framework, as well as indifference/uncertainty. Because preference can be expressed over arbitrary subsets of the frame this is in fact a multi-polar model for expressing and combining preferences. Even in the case of no overlapping focal elements the belief constraint operator provides a meaningful answer, namely that the preferences are incompatible.

4.9.2 Examples

Expressing Preferences with Subjective Opinions

Preferences can be expressed e.g. as soft or hard constraints, qualitative or quantitative, ordered or partially ordered etc. It is possible to specify a mapping between qualitative verbal tags and subjective opinions which enables easy solicitation of preferences [22]. Table 4.8 describes examples of how preferences can be expressed.

All the preference types of Table 4.8 can be interpreted in terms of subjective opinions, and further combined by considering them as constraints expressed by different agents. The examples that comprise two binary frames could also have been modelled with a quaternary product frame with a corresponding 4-nomial product opinion. In fact product opinions over product frames could be a method of simultaneously considering preferences over multiple variables, and this will be the topic of future research.

Default base rates are specified in all but the last example which indicates total indifference but with a bias which expresses the average preference in the population. Base rates are useful in many situations, such as for default reasoning. Base rates only have an influence in case of significant indifference or uncertainty.

Going to the Cinema, 1st Attempt

Assume three friends, Alice, Bob and Clark, who want to see a film together at the cinema one evening, and that the only films showing are *Black Dust* (*BD*), *Grey Matter* (*GM*) and *White Powder* (*WP*), represented as the ternary frame $\Theta = \{BD,\ GM,\ WP\}$. Assume that the friends express their preferences in the form of the opinions of Table 4.9.

Alice and Bob have strong and conflicting preferences. Clark, who only does not want to watch *Black Dust*, and who is indifferent about the two other films, is not sure whether he wants to come along, so Table 4.9 shows the results of applying the preference combination operator, first without him, and then including in the party.

By applying the belief constraint operator Alice and Bob conclude that the only film they are both interested in seeing is *Grey Matter*. Including Clark in the party does not change that result because he is indifferent to *Grey Matter* and *White Powder* anyway, he just does not want to watch the film *Black Dust*.

The belief mass values of Alice and Bob in the above example are in fact equal to those of Zadeh's example [26] which was used to demonstrate the unsuitability of Dempster's rule for fusing beliefs because it produces

Example & Type	Opinion Expression	
"Ingredient x is mandatory"	Binary frame	$X = \{x, \overline{x}\}$
Hard positive	Binomial opinion	$\omega_x : (1, 0, 0, \frac{1}{2})$
"Ingredient x is totally out of the question"	Binary frame	$X = \{x, \overline{x}\}$
Hard negative	Binomial opinion	$\omega_x : (0,\ 1,\ 0,\ \frac{1}{2})$
"My preference rating for x is 3 out of 10	Binary frame	$X = \{x, \overline{x}\}$
Quantitative	Binomial opinion	$\omega_x : (0.3,\ 0.7,\ 0.0,\ \frac{1}{2})$
"I prefer x or y, but z is also acceptable"	Ternary frame	$\Theta = \{x, y, z\}$
Qualitative	Trinomial opinion	$\omega_\Theta : (b(x,y) = 0.6,\ b(z) = 0.3,$ $u = 0.1,\ a(x,y,z) = \frac{1}{3})$
"I like x, but I like y even more"	Two binary frames	$X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$
Positive rank	Binomial opinions	$\omega_x : (0.6,\ 0.3,\ 0.1,\ \frac{1}{2})$, $\omega_y : (0.7,\ 0.2,\ 0.1,\ \frac{1}{2})$
"I don't like x, and I dislike y even more"	Two binary frames	$X = \{x, \overline{x}\}$ and $Y = \{y, \overline{y}\}$
Negative rank	Binomial opinions	$\omega_x : (0.3,\ 0.6,\ 0.1,\ \frac{1}{2})$, $\omega_y : (0.2,\ 0.7,\ 0.1,\ \frac{1}{2})$
"I'm indifferent about x, y and z"	Ternary frame	$\Theta = \{x, y, z\}$
Neutral	Trinomial opinion	$\omega_\Theta : (u_\Theta = 1.0,\ a(x,y,z) = \frac{1}{3})$
"I'm indifferent but most people prefer x"	Ternary frame	$\Theta = \{x, y, z\}$
Neutral with bias	Trinomial opinion	$\omega_\Theta : (u_\Theta = 1.0,\ a(x) = 0.6,$ $a(y) = 0.2, a(z) = 0.2)$

Table 4.8: Example preferences and corresponding subjective opinions

		Preferences of:			Results of preference combinations:	
		Alice	Bob	Clark	(Alice & Bob)	(Alice & Bob & Clark)
		ω^A_Θ	ω^B_Θ	ω^C_Θ	$\omega^{A\&B}_\Theta$	$\omega^{A\&B\&C}_\Theta$
$b(BD)$	=	0.99	0.00	0.00	0.00	0.00
$b(GM)$	=	0.01	0.01	0.00	1.00	1.00
$b(WP)$	=	0.00	0.99	0.00	0.00	0.00
$b(GM \cup WP)$	=	0.00	0.00	1.00	0.00	0.00

Table 4.9: Combination of film preferences

counterintuitive results. Zadeh's example describes a medical case where two medical doctors express their opinions about possible diagnoses, which typically should have been modelled with the averaging fusion operator [11], not with Dempster's rule. In order to select the appropriate operator it is crucial to fully understand the nature of the situation to be modelled. The failure to understand that Dempster's rule does not represent an operator for cumulative or averaging belief fusion, combined with the unavailability of the general cumulative and averaging belief fusion operators for many years (1976[24]-2010[11]), has often led to inappropriate applications of Dempster's rule to cases of belief fusion [14]. However, when specifying the same numerical values as in [26] in a case of preference combination such as the example above, the belief constraint operator which is a simple extension of Dempster's rule is very suitable and produces perfectly intuitive results.

Going to the Cinema, 2nd Attempt

In this example Alice and Bob soften their strong preference by expressing some indifference in the form of $u = 0.01$, as specified by Table 4.10. Clark has the same opinion as in the previous example, and is still not sure whether he wants to come along, so Table 4.10 shows the results without and with his preference included.

The effect of adding some indifference is that Alice and Bob should pick film *Black Dust* or *White Powder* because in both cases one of them actually prefers one of the films, and the other finds it acceptable. Neither Alice

		Preferences of:			Results of preference combinations:	
		Alice	Bob	Clark	(Alice & Bob)	(Alice & Bob & Clark)
		ω_Θ^A	ω_Θ^B	ω_Θ^C	$\omega_\Theta^{A\&B}$	$\omega_\Theta^{A\&B\&C}$
$b(BD)$	=	0.98	0.00	0.00	0.490	0.000
$b(GM)$	=	0.01	0.01	0.00	0.015	0.029
$b(WP)$	=	0.00	0.98	0.00	0.490	0.961
$b(GM \cup WP)$	=	0.00	0.00	1.00	0.000	0.010
u	=	0.01	0.01	0.00	0.005	0.000
$a(BD)$	=	0.6	0.6	0.6	0.6	0.6
$a(GM) = a(WP)$	=	0.2	0.2	0.2	0.2	0.2

Table 4.10: Combination of film preferences with some indifference and with non-default base rates

nor Bob prefers *Gray Matter*, they only find it acceptable, so it turns out not to be a good choice for any of them. When taking into consideration that the base rate $a(BD) = 0.6$ and the base rate $a(WP) = 0.2$, the preference expectation values according to Eq.(3.14) are such that:

$$\mathrm{E}^{A\&B}(BD) > \mathrm{E}^{A\&B}(WP)\ . \tag{4.105}$$

More precisely, the preference expectation values according to Eq.(3.14) are:

$$\mathrm{E}^{A\&B}(BD) = 0.493\ , \qquad \mathrm{E}^{A\&B}(WP) = 0.491\ . \tag{4.106}$$

Because of the higher base rate, *Black Dust* also has a higher expected preference than *White Powder*, so the rational choice would be to watch *Black Dust*.

However, when including Clark who does not want to watch *Black Dust*, the base rates no longer dictates the result. In this case Eq.(3.14) produces $\mathrm{E}^{A\&B\&C}(WP) = 0.966$ so the obvious choice is to watch *White Powder*.

Not Going to the Cinema

Assume now that the Alice and Bob express totally conflicting preferences as specified in Table 4.11, i.e. Alice expresses a hard preference for *Black Dust* and Bob expresses a hard preference for *White Powder*. Clark still has the same preference as before, i.e he does not want to watch *Black Dust* and is indifferent about the two other films.

		Preferences of:			Results of preference combinations:	
		Alice	Bob	Clark	(Alice & Bob)	(Alice & Bob & Clark)
		ω_Θ^A	ω_Θ^B	ω_Θ^C	$\omega_\Theta^{A\&B}$	$\omega_\Theta^{A\&B\&C}$
$b(BD)$	=	1.00	0.00	0.00	Undefined	Undefined
$b(GM)$	=	0.00	0.00	0.00	Undefined	Undefined
$b(WP)$	=	0.00	1.00	0.00	Undefined	Undefined
$b(GM \cup WP)$	=	0.00	0.00	1.00	Undefined	Undefined

Table 4.11: Combination of film preferences with hard and conflicting preferences

In this case the belief constraint operator can not be applied because Eq.(4.102) produces a division by zero. The conclusion is that the friends will not go to the cinema to see a film together. The test for detecting this situation is when $Con = 1$ in Eq.(4.103). It makes no difference to include Clark in the party because a conflict can not be resolved by including additional preferences. However it would have been possible for Bob and Clark to watch *White Powder* together without Alice.

Chapter 5

Applications

Subjective logic represents a generalisation of probability calculus and logic under uncertainty. Subjective logic will always be equivalent to traditional probability calculus when applied to traditional probabilities, and will be equivalent to binary logic when applied to TRUE and FALSE statements.

While subjective logic has traditionally been applied to binary frames, we have shown that it can easily be extended and be applicable to frames larger than binary. The input and output parameters of subjective logic are beliefs in the form of opinions. We have described three different equivalent notations of opinions which provides rich interpretations of opinions. This also allows the analyst to choose the opinion representation that best suits a particular situation.

5.1 Fusion of Opinions

The cumulative and averaging rules of belief fusion make it possible to use the theory of belief functions for modelling situations where evidence is combined in a cumulative or averaging fashion. Such situations could previously not be correctly modelled within the framework of belief theory. It is worth noticing that the cumulative, averaging rules and Dempster's rule apply to different types of belief fusion, and that, strictly speaking, is meaningless to compare their performance in the same examples. The notion of cumulative and averaging belief fusion as opposed to conjunctive belief fusion has therefore been introduced in order to make this distinction explicit.

The following scenario will illustrate using the cumulative and the averaging fusion operators, as well as multiplication. Assume that a GE (Genetical Engineering) process can produce Male (M) or Female (F) eggs, and that in addition, each egg can have genetical mutation X or Y independently of its gender. This constitutes the quaternary frame $\Theta = \{MX, MY, FX, FY\}$. Sensors IA and IB simultaneously observe whether each egg is M or F, and Sensor II observes whether the egg has mutation X or Y.

Assume that Sensors IA and IB have derived two separate opinions regarding the gender of a specific egg, and that Sensor II has produced an opinion regarding its mutation. Because Sensors IA and IB have observed the same aspect simultaneously, the opinions should be fused with averaging fusion. Sensor II has observed a different and orthogonal aspect, so the output of the averaging fusion and the opinion of Sensor II should be combined with multiplication. This is illustrated in Fig.5.1.

This result from fusing the two orthogonal opinions with multiplication can now be considered as a single observation. By combining opinions from multiple observations it is possible to express the most likely status of future eggs as a predictive opinion. We are now dealing with two different situations which must be considered separately. The first situation relates to the state of a given egg that the sensors have already observed. The second situation relates to the possible state of eggs that will be produced in the future. An opinion in the first situation is based on the sensors as illustrated inside Observation 1 in Fig.5.1. The second situation relates to combining multiple observations, as illustrated by fusing the opinions from Observation 1 and Observation 2 in Fig.5.1.

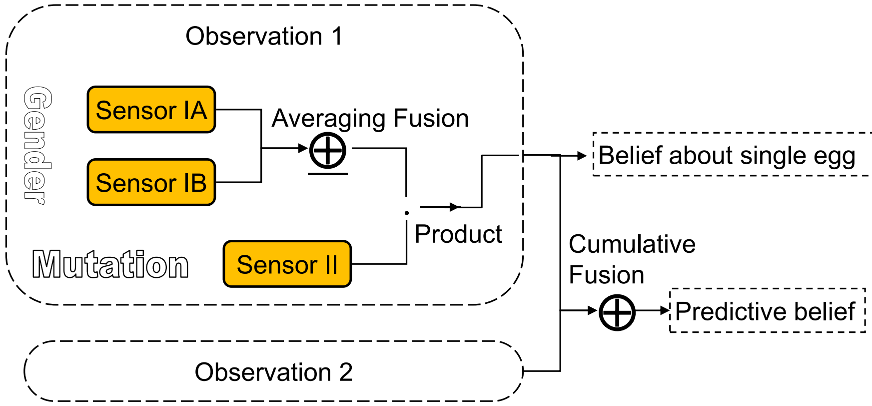


Figure 5.1: Applying different types of belief fusion according to the situation

5.2 Bayesian Networks with Subjective Logic

A Bayesian network is a graphical model for conditional relationships. Specifically, a Bayesian network is normally defined as a directed acyclic graph of nodes representing variables and arcs representing conditional dependence relations among the variables.

Equipped with the operators for conditional deduction and abduction, it is possible to analyse Bayesian networks with subjective logic. For example, the simple Bayesian network:

$$X \longrightarrow Y \longrightarrow Z \tag{5.1}$$

can be modelled by defining conditional opinions between the three frames. In case conditionals can be obtained with X as antecedent and Y as consequent, then deductive reasoning can be applied to the edge $[X : Y]$. In case there are available conditionals with Y as antecedent and X as consequent, then abductive reasoning must be applied.

In the example illustrated in Fig.5.2 it is assumed that deductive reasoning can be applied to both $[X : Y]$ and $[Y : Z]$.

The frames X and Y thus represent parent and child of the first conditional edge, and the frames Y and Z represent parent and child of the second conditional edge respectively.

This chaining of conditional reasoning is possible because of the symmetry between the parent and child frames. They both consist of sets of mutually exclusive elements, and subjective opinions can be applied to both. In general it is arbitrary which frame is the antecedent and which frame is the consequent in a given conditional edge. Conditional reasoning is possible in either case, by applying the deductive or the abductive operator.

When there is a degree of relevance between a parent and child frame pair, this relevance can be expressed as a conditional parent-child relationship. The relevance is directional, and is directly reflected by the conditionals. For probabilistic conditional deduction from x to y the relevance of x to y can be defined as:

$$\mathrm{R}(y|x) = |p(y|x) - p(y|\overline{x})| \,. \tag{5.2}$$

It can be seen that $\mathrm{R}(y|x) \in [0, 1]$, where $\mathrm{R}(y|x) = 0$ expresses total irrelevance, and $\mathrm{R}(y|x) = 1$ expresses total relevance between y and x.

For conditionals expressed as opinions, the same type of relevance between a given state $y_j \in Y$ and a given state $x_i \in X$ can be defined as:

$$\mathrm{R}(y_j|x_i) = |\mathrm{E}(\omega_{Y|x_i}(y_j) - \mathrm{E}(\omega_{Y|\overline{x_j}}(y_j))| \,. \tag{5.3}$$

The relevance of a parent state $x_i \in X$ to a child frame Y can be defined as:

$$\mathrm{R}(Y|x_i) = \sum_{j=1}^{l} \mathrm{R}(y_j|x_i)/l \,. \tag{5.4}$$

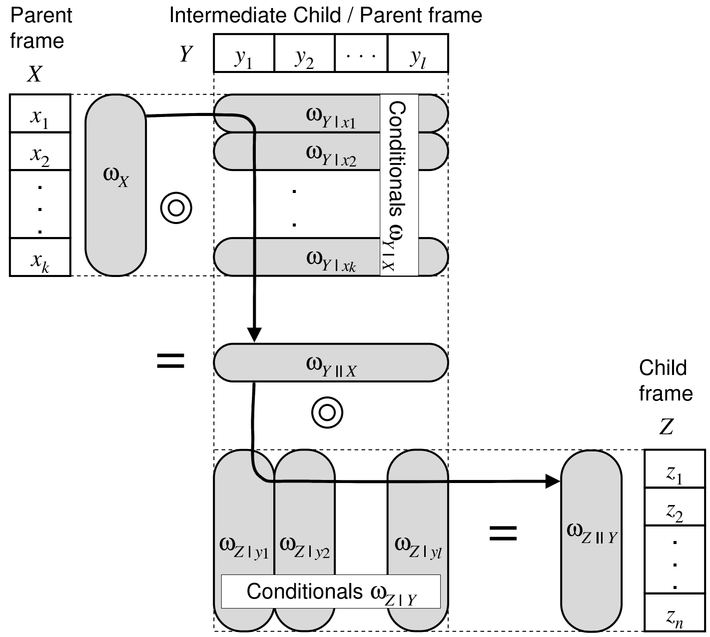


Figure 5.2: Deductive opinion structure for the Bayesian network of Eq.(5.1)

Finally, the relevance of a parent frame X to a child frame Y can be expressed as:

$$\mathrm{R}(Y|X) = \sum_{i=1}^{k} \mathrm{R}(Y|x_i)/k\ . \tag{5.5}$$

In our model, the relevance measure of Eq.(5.5) is the most general.

In many situations there can be multiple parents for the same child, which requires fusion of the separate child opinions into a single opinion. The question then arises which type of fusion is most appropriate. The two most typical situations to modelled are the cumulative case and the averaging case.

Cumulative fusion is applicable when independent evidence is accumulated over time such as by continuing observation of outcomes of a process. Averaging fusion is applicable when two sources provide different but independent opinions so that each opinion is weighed as a function of its certainty. The fusion operators are described in Sec.4.7.

By fusing child opinions resulting from multiple parents, arbitrarily large Bayesian networks can be constructed. Depending on the situation it must be decided whether the cumulative or the averaging operator is applicable. An example with three grandparent frames X_1, X_2, X_3, two parent parent frames Y_1, Y_2 and one child frame Z is illustrated in Fig.5.3 below.

The nodes X_1, X_2, X_3 and Y_2 represent initial parent frames because they do not themselves have parents in the model. Opinions about the initial parent nodes represent the input evidence to the model.

When representing Bayesian networks as graphs, the structure of conditionals is hidden in the edges, and only the nodes consisting of parent and children frames are shown.

When multiple parents can be identified for the same child, there are two important considerations. Firstly, the relative relevance of each parent to each child, and secondly the relevance or dependence between the parents.

Strong relevance of parent to child frames is desirable, and models should strive to include the strongest parent-child relationships that can be identified, and for which there is evidence directly or potentially available.

Dependence between parents should be avoided as far as possible. A more subtle and hard to detect dependence can originate from hidden parent nodes outside the Bayesian network model itself. In this case the parent nodes have a hidden common grand parent node which makes them dependent. Philosophically speaking everything depends on everything in some way, so absolute independence is never achievable. There will often be some degree of dependence between evidence sources, but which from a practical perspective can be ignored. When

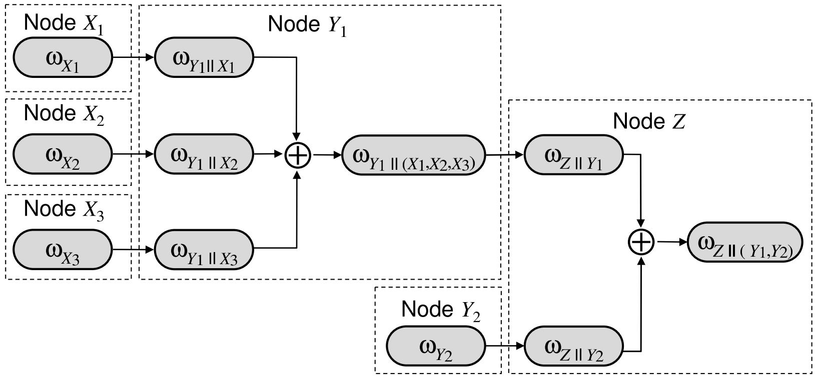


Figure 5.3: Bayesian network with multiple parent evidence nodes

building Bayesian network models it is important to be aware of possible dependencies, and try to select parent evidence nodes that have the lowest possible degree of dependence.

As an alternative method for managing dependence it could be possible to allow different children to share the same parent by fissioning the parent opinion, or alternatively taking dependence into account during the fusion operation. The latter option can be implemented by applying the averaging fusion operator.

It is also possible that evidence opinions provided by experts need to be discounted due to the analysts doubt in their reliability. This can be done with the trust transitivity operator[1] described in Sec.4.8

[1]Also called discounting operator

Bibliography

[1] Bruno de Finetti. The true subjective probability problem. In Carl-Axel Staël von Holstein, editor, *The concept of probability in psychological experiments*, pages 15–23, Dordrecht, Holland, 1974. D.Reidel Publishing Company.

[2] M. Fitting. Kleene's three-valued logics and their children. *Fundamenta Informaticae*, 20:113–131, 1994.

[3] A. Gelman et al. *Bayesian Data Analysis, 2nd ed*. Chapman and Hall/CRC, Florida, USA, 2004.

[4] Robin K.S. Hankin. A generalization of the dirichlet distribution. *Journal of Statistical Software*, 33(11):1–18, February 2010.

[5] U. Hoffrage, S. Lindsey, R. Hertwig, and G. Gigerenzer. Communicating statistical information. *Science*, 290(5500):2261–2262, December 2000.

[6] A. Jøsang. A Logic for Uncertain Probabilities. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9(3):279–311, June 2001.

[7] A. Jøsang. The Consensus Operator for Combining Beliefs. *Artificial Intelligence*, 142(1–2):157–170, October 2002.

[8] A. Jøsang. Abductive Reasoning with Uncertainty. In *The Proceedings of the International Conference on Information Processing and Management of Uncertainty (IPMU2008)*, Malaga, June 2008.

[9] A. Jøsang. Conditional Reasoning with Subjective Logic. *Journal of Multiple-Valued Logic and Soft Computing*, 15(1):5–38, 2008.

[10] A. Jøsang. Cumulative and Averaging Unfusion of Beliefs. In *The Proceedings of the International Conference on Information Processing and Management of Uncertainty (IPMU2008)*, Malaga, June 2008.

[11] A. Jøsang, J. Diaz, and M. Rifqi. Cumulative and Averaging Fusion of Beliefs. *Information Fusion*, 11(2):192–200, 2010. doi:10.1016/j.inffus.2009.05.005.

[12] A. Jøsang and S.J. Knapskog. A Metric for Trusted Systems (full paper). In *Proceedings of the 21st National Information Systems Security Conference*. NSA, October 1998.

[13] A. Jøsang and D. McAnally. Multiplication and Comultiplication of Beliefs. *International Journal of Approximate Reasoning*, 38(1):19–51, 2004.

[14] A. Jøsang and S. Pope. Dempster's Rule as Seen by Little Colored Balls. *Computational Intelligence*, 28(4), November 2012.

[15] A. Jøsang, S. Pope, and S. Marsh. Exploring Different Types of Trust Propagation. In *Proceedings of the 4th International Conference on Trust Management (iTrust)*, Pisa, May 2006.

[16] Audun Jøsang. Multi-Agent Preference Combination using Subjective Logic. In *International Workshop on Preferences and Soft Constraints (Soft'11)*, Perugia, Italy, 2011.

[17] Audun Jøsang and Robin Hankin. Interpretation and Fusion of Hyper Opinions in Subjective Logic. In *Proceedings of the 15th International Conference on Information Fusion (FUSION 2012)*, Singapore, July 2012.

[18] Audun Jøsang, Simon Pope, and Milan Daniel. Conditional deduction under uncertainty. In *Proceedings of the 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005)*, 2005.

[19] Sherman Kent. Words of estimated probability. In Donald P. Steury, editor, *Sherman Kent and the Board of National Estimates: Collected Essays*. CIA, Center for the Study of Intelligence, 1994.

[20] D. McAnally and A. Jøsang. Addition and Subtraction of Beliefs. In *Proceedings of Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2004)*, Perugia, July 2004.

[21] N. J. Nilsson. Probabilistic logic. *Artificial Intelligence*, 28(1):71–87, 1986.

[22] Simon Pope and Audun Jøsang. Analsysis of Competing Hypotheses using Subjective Logic. In *Proceedings of the 10th International Command and Control Research and Technology Symposium (ICCRTS)*. United States Department of Defense Command and Control Research Program (DoDCCRP), 2005.

[23] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev. *Integrals and Series*, (translated from Russian), volume 1–3. Gordon and Breach Science Publishers, Amsterdam, New York, 1986.

[24] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[25] M. Smithson. *Ignorance and Uncertainty: Emerging Paradigms*. Springer, 1988.

[26] L.A. Zadeh. Review of Shafer's A Mathematical Theory of Evidence. *AI Magazine*, 5:81–83, 1984.